

1. **Definition.**

Let A be a set. A is said to be **finite** if there exists some $n_A \in \mathbb{N}$ such that $A \sim \llbracket 1, n_A \rrbracket$. The number n_A is called the cardinality of A .

Remark. The number n_A is uniquely determined by A . (Proof?)

2. **Illustration of properties of finite sets through an example.**

Consider the set $\{1, 2, 3\}$.

(1) *Question.*

Is there some injective function from \mathbb{N} to $\{1, 2, 3\}$? Is there an infinite sequence with no repeated terms in $\{1, 2, 3\}$?

Answer and reason.

No. Were there an infinite sequence $(a_n)_{n=0}^{\infty}$ with no repeated terms in $\{1, 2, 3\}$, we would have $\{a_0, a_1, a_2\} = \{1, 2, 3\}$ and then a_3 would have to be one of a_0, a_1, a_2 .

(2) *Question.*

Is there some proper subset U of $\{1, 2, 3\}$ satisfying $\{1, 2, 3\} \sim U$?

Answer and reason.

No. (Heuristic argument.) Every proper subset of $\{1, 2, 3\}$ has at most two elements, but $\{1, 2, 3\}$ has three elements. They are not ‘of the same size’.

(3) *Question.*

Is there some function $\varphi : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ which is injective and not surjective?

Answer and reason.

No. There are six injective functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$. They are given by:

$$\begin{array}{ll} \varphi_1(1) = 1, \varphi_1(2) = 2, \varphi_1(3) = 3. & \varphi_4(1) = 1, \varphi_4(2) = 3, \varphi_4(3) = 2. \\ \varphi_2(1) = 2, \varphi_2(2) = 3, \varphi_2(3) = 1. & \varphi_5(1) = 2, \varphi_5(2) = 1, \varphi_5(3) = 3. \\ \varphi_3(1) = 3, \varphi_3(2) = 1, \varphi_3(3) = 2. & \varphi_6(1) = 3, \varphi_6(2) = 2, \varphi_6(3) = 1. \end{array}$$

They are all surjective.

(4) *Question.*

Is there some function $\psi : \{1, 2, 3\} \rightarrow \{1, 2, 3\}$ which is surjective and not injective?

Answer and reason.

No. There are six surjective functions from $\{1, 2, 3\}$ to $\{1, 2, 3\}$. They are given by:

$$\begin{array}{ll} \psi_1(1) = 1, \psi_1(2) = 2, \psi_1(3) = 3. & \psi_4(1) = 1, \psi_4(2) = 3, \psi_4(3) = 2. \\ \psi_2(1) = 2, \psi_2(2) = 3, \psi_2(3) = 1. & \psi_5(1) = 2, \psi_5(2) = 1, \psi_5(3) = 3. \\ \psi_3(1) = 3, \psi_3(2) = 1, \psi_3(3) = 2. & \psi_6(1) = 3, \psi_6(2) = 2, \psi_6(3) = 1. \end{array}$$

They are all injective.

We can ask the same questions for every other finite set, and obtain the same answers along the same line of reasoning.

3. **Theorem (XIX). (Characterization of finite sets.)**

Let A be a set. The statements below are equivalent:

- (1) A is finite.
- (2) No proper subset of A is of cardinality equal to A .
- (3) For any function φ from A to A , if φ is injective then φ is surjective.
- (4) For any function ψ from A to A , if ψ is surjective then ψ is injective.

Proof of Theorem (XIX). A very tedious exercise in mathematical induction.

4. Definition.

Let A be a set. A is said to be **infinite** if $\mathbb{N} \lesssim A$.

Remark. Heuristic idea in this definition: A is infinite iff A contains at least a ‘copy’ of \mathbb{N} as a subset.

Examples. $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, [0, 1]$, line segments, circles, squares, discs in the plane \mathbb{R}^2 , ...

5. Illustration of properties of infinite sets through an example.

Consider the set $[0, 1]$.

(1) *Question.*

Is there some injective function from \mathbb{N} to $[0, 1]$? Is there an infinite sequence with no repeated terms in $[0, 1]$?

Answer and reason.

Yes. One such infinite sequence is $\{1/2^n\}_{n=0}^{\infty}$. So an injective function g from \mathbb{N} to $[0, 1]$ is given by $g(n) = 1/2^n$ for any $n \in \mathbb{N}$.

(2) *Question.*

Is there some proper subset U of $[0, 1]$ satisfying $[0, 1] \sim U$?

Answer and reason.

Yes. One such set is $[0, 1)$.

(3) *Question.*

Is there some function $\varphi : [0, 1] \rightarrow [0, 1]$ which is injective and not surjective?

Answer and reason.

Yes. One such function is given by $\varphi : [0, 1] \rightarrow [0, 1]$, where $S = \{1/2^n \mid n \in \mathbb{N}\}$ and

$$\varphi(x) = \begin{cases} x & \text{if } x \in [0, 1] \setminus S \\ x/2 & \text{if } x \in S \end{cases}$$

(Note that for any $x \in [0, 1]$, we have $\varphi(x) \neq 1$.)

(4) *Question.*

Is there some function $\psi : [0, 1] \rightarrow [0, 1]$ which is surjective and not injective?

Answer and reason.

Yes. One such function is given by $\psi : [0, 1] \rightarrow [0, 1]$, where $S = \{1/2^n \mid n \in \mathbb{N}\}$ and

$$\psi(x) = \begin{cases} x & \text{if } x \in [0, 1] \setminus S \\ 2x & \text{if } x \in S \setminus \{1\} \\ 1 & \text{if } x = 1 \end{cases}$$

(Note that we have $\psi(1) = \psi(1/2) = 1$.)

We can ask the same questions for every other infinite set, and obtain the same answers along the same line of reasoning.

6. Theorem (XX). (Characterization of infinite sets.)

Let A be a set. The statements below are equivalent:

- (1) A is infinite. ($\mathbb{N} \lesssim A$.)
- (1') There exists some subset S of A such that $\mathbb{N} \sim S$.
- (1'') There exists some subset T of A such that $\mathbb{N} \lesssim T$.
- (2) There exists some proper subset U of A such that $A \sim U$.
- (2') There exists some proper subset V of A such that $A \lesssim V$.
- (3) There exists some function φ from A to A such that φ is injective and φ is not surjective.
- (4) There exists some function ψ from A to A such that ψ is surjective and ψ is not injective.

Remark. By Theorem (XIX) and Theorem (XX), every set is finite or infinite, but not both.

7. Proof of Theorem (XX).

Let A be a set.

(a) Suppose A is infinite.

[We want to deduce that there exists some proper subset U of A such that $A \sim U$.]

Pick some injective function $g : \mathbb{N} \rightarrow A$.

Define $z = g(0)$. Define $U = A \setminus \{z\}$.

Note that U is a proper subset of A .

Define the relation $h = (A, U, H)$ by

$$H = \{(x, x) \mid x \in A \setminus g(\mathbb{N})\} \cup \{(g(n), g(n+1)) \mid n \in \mathbb{N}\}$$

h is a bijective function from A to U . Hence $A \sim U$.

[Note that the function $\text{id}_{A \setminus g(\mathbb{N})}$ is bijective. Also note that the function from $g(\mathbb{N})$ to $g(\mathbb{N}) \setminus \{z\}$ given by $n \mapsto n+1$ for any $n \in \mathbb{N}$ is bijective. The bijective function h is obtained by glueing these two bijective functions according the Glueing Lemma.]

(b) Suppose there exists some proper subset U of A such that $A \sim U$.

[We want to deduce that there exists some function φ from A to A such that φ is injective and φ is not surjective.]

Pick some bijective function $h : A \rightarrow U$.

Denote by $\iota_U : U \rightarrow A$ by $\iota_U(x) = x$ for any $x \in U$.

Define the function $\varphi : A \rightarrow A$ by $\varphi = \iota_U \circ h$.

Since h, ι_U are both injective, φ is also injective.

Note that $\varphi(A) = (\iota_U \circ h)(A) = \iota_U(h(A)) = \iota_U(U) = U \subsetneq A$. Then φ is not surjective.

(c) Suppose there exists some function φ from A to A such that φ is injective and φ is not surjective.

[We want to deduce that A is infinite according to definition. We are going to construct an infinite sequence in A with no repeated terms.]

Since φ is not surjective, $A \setminus \varphi(A) \neq \emptyset$.

Pick some $b \in A \setminus \varphi(A)$. Define $b_0 = b$. For any $m \in \mathbb{N} \setminus \{0\}$, we define $b_m = \varphi(b_{m-1})$.

By definition, for each $n \in \mathbb{N}$ we have $b_n = \underbrace{(\varphi \circ \varphi \circ \dots \circ \varphi \circ \varphi)}_{n \text{ times}}(b)$. (Why? Apply mathematical induction.)

Then, by the injectivity of φ , we have $b_j \neq b_k$ whenever $j \neq k$. (Why?)

It follows that the function $f : \mathbb{N} \rightarrow A$ defined by $f(n) = b_n$ for any $n \in \mathbb{N}$ is an injective function.

Therefore $\mathbb{N} \lesssim A$. Hence A is infinite.

(d) Suppose there exists some function φ from A to A such that φ is injective and φ is not surjective.

[We want to deduce that there exists some function ψ from A to A such that ψ is surjective and ψ is not injective.]

Note that $A \neq \emptyset$. (Why?)

Define the function $\hat{\varphi} : A \rightarrow \varphi(A)$ by $\hat{\varphi}(x) = \varphi(x)$ for any $x \in A$.

By the injectivity of φ , the function $\hat{\varphi}$ is injective.

By the definition of $\hat{\varphi}$, we have $\hat{\varphi}(A) = \varphi(A)$. Then $\hat{\varphi}$ is surjective.

Therefore $\hat{\varphi}$ is a bijective function.

Since φ is not surjective, $A \setminus \varphi(A) \neq \emptyset$. Pick some $z_0 \in A \setminus \varphi(A)$.

Define the function $\psi : A \rightarrow A$ by

$$\psi(y) = \begin{cases} \hat{\varphi}^{-1}(y) & \text{if } y \in \varphi(A) \\ z_0 & \text{if } y \in A \setminus \varphi(A) \end{cases}$$

We have $\psi(A) \supset \psi(\varphi(A)) = \psi(\hat{\varphi}(A)) = \hat{\varphi}^{-1}(\hat{\varphi}(A)) = A$.

Then ψ is surjective.

We verify that ψ is not injective:

- Since $z_0 \in A \setminus \varphi(A)$, we have $z_0 \neq \varphi(z_0)$.
We have $\psi(z_0) = z_0$ and $\psi(\varphi(z_0)) = \hat{\varphi}^{-1}(\hat{\varphi}(z_0)) = z_0$.
Then $\psi(z_0) = \psi(\varphi(z_0))$.
Therefore ψ is not injective.

(e) Suppose there exists some function ψ from A to A such that ψ is surjective and ψ is not injective.

[We want to deduce that there exists some function φ from A to A such that φ is injective and φ is not surjective.]

By the surjectivity of ψ , for any $x \in A$, there exists some $y_x \in A$ such that $\psi(y_x) = x$. [Here we need the Axiom of Choice.]

Define the function $\varphi : A \rightarrow A$ by $\varphi(x) = y_x$ for any $x \in A$.

We verify that φ is injective:

- Pick any $x, x' \in A$. Suppose $\varphi(x) = \varphi(x')$. Then $y_x = y_{x'}$. Therefore $x = \psi(y_x) = \psi(y_{x'}) = x'$.
So φ is injective.

We verify that φ is not surjective:

- Since ψ is not injective, there exist some $x_0, z_0 \in A$ such that $x_0 \neq z_0$ and $\psi(x_0) = \psi(z_0)$.
By the definition of φ , we have $\varphi(\psi(x_0)) = x_0$ and $\varphi(\psi(z_0)) = z_0$. Then by the definition of φ , we have $z_0 \notin \varphi(A)$.
So φ is not surjective.