1. Recall:

   (a) **Definition.**

   Let A, B be sets.

   The set $\mathsf{Map}(A, B)$ *is defined to be the* **set of all functions from A to B.**

   Remark. $\mathsf{Map}(\mathbb{N}, B)$ is the set of all infinite sequences in B: each $\varphi \in \mathsf{Map}(\mathbb{N}, B)$ is the infinite sequence $(\varphi(0), \varphi(1), \varphi(2), ..., \varphi(n), \varphi(n+1), ...)$.

   (b) **Example** (ϵ).

   Let A be a set. $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0, 1\})$.

   (c) **Theorem (VI).**

   There is no surjective function from $\mathbb{N}$ *to* $\mathsf{Map}(\mathbb{N}, \{0, 1\})$.

   (d) **Corollary (VII).**

   There is no bijective function from $\mathbb{N}$ *to* $\mathsf{Map}(\mathbb{N}, \{0, 1\})$. *(Hence* $\mathbb{N} \not\sim \mathsf{Map}(\mathbb{N}, \{0, 1\})$.*)*

   (e) **Theorem (VIII)**.

   Let A be a set. $A \not\sim \mathsf{Map}(A, \{0, 1\})$. $A \not\sim \mathfrak{P}(A)$.

2. **Theorem (XIII). (Baby version of Cantor's Theorem.)**

$\mathbb{N} < \mathsf{Map}(\mathbb{N}, \{0,1\})$.

Proof. [Want to verify: ① $\mathbb{N} \not\sim \mathsf{Map}(\mathbb{N}, \{0,1\})$. ② $\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0,1\})$.]

By Corollary (VII), $\mathbb{N} \not\sim \mathsf{Map}(\mathbb{N}, \{0,1\})$.

We now prove that $\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0,1\})$:

- For any $n \in \mathbb{N}$, define $\delta_n : \mathbb{N} \longrightarrow \{0,1\}$ by

$$\delta_n(k) = \begin{cases} 1 & \text{if} \quad k = n \\ 0 & \text{if} \quad k \neq n \end{cases}$$

The infinite sequence δ_n is explicitly given by

$\delta_n = (\underbrace{0, \ldots, 0}_{\text{all 0's}}, \underset{n^{th} \text{ position}}{1}, \underbrace{0, 0, \ldots}_{\text{all 0's}})$.

Define $\Delta : \mathbb{N} \longrightarrow \mathsf{Map}(\mathbb{N}, \{0,1\})$ by $\Delta(n) = \delta_n$ for any $n \in \mathbb{N}$.

Δ is an injective function. (Why?)

Hence $\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0,1\})$.

We now have $\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0,1\})$ and $\mathbb{N} \not\sim \mathsf{Map}(\mathbb{N}, \{0,1\})$.

It follows that $\mathbb{N} < \mathsf{Map}(\mathbb{N}, \{0,1\})$.

3. **Theorem (XIV). (Cantor's Theorem.)**

Suppose A is a set. Then $A < \mathsf{Map}(A, \{0,1\})$, and $A < \mathfrak{P}(A)$.

Proof.

Let A be a set. By Theorem (VIII), $A \not\sim \mathsf{Map}(A, \{0,1\})$.

We prove that $A \lesssim \mathsf{Map}(A, \{0,1\})$:

- Recall that for any $x \in A$, the function $\chi^A_{\{x\}} : A \longrightarrow \{0,1\}$ is given by

  $$\chi^A_{\{x\}}(y) = \begin{cases} 1 & \text{if} \quad y = x \\ 0 & \text{if} \quad y \neq x \end{cases}$$

  Define the function $\Delta : A \longrightarrow \mathsf{Map}(A, \{0,1\})$ by $\Delta(x) = \chi^A_{\{x\}}$ for any $x \in A$.
  Δ is an injective function from A to $\mathsf{Map}(A, \{0,1\})$. (Why?)
  Hence $A \lesssim \mathsf{Map}(A, \{0,1\})$.

We now have $A \lesssim \mathsf{Map}(A, \{0,1\})$ and $A \not\sim \mathsf{Map}(A, \{0,1\})$.
It follows that $A < \mathsf{Map}(A, \{0,1\})$.
Since $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0,1\})$, we have $A < \mathfrak{P}(A)$. (Why?)

4. **Question.** *Note that $\mathbb{Q} \lesssim \mathbb{R}$. Is it true that $\mathbb{Q} \sim \mathbb{R}$, or that $\mathbb{Q} < \mathbb{R}$?*

Lemma (XV).

Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Also suppose $A < B$ or $B < C$. Then $A < C$.

Proof. [Needed in the argument: Schröder-Bernstein Theorem. 'Let H, K be sets. Suppose $H \lesssim K$ and $K \lesssim H$. Then $H \sim K$.']

Let A, B, C be sets. Suppose $A \lesssim B$ and $B \lesssim C$. Also suppose $A < B$ or $B < C$.

Since $A \lesssim B$ and $B \lesssim C$, we have $A \lesssim C$.

Since $A < B$ or $B < C$, we have $A \not\sim B$ or $B \not\sim C$. We verify that $A \not\sim C$:

- Suppose it were true that $A \sim C$. Then $C \lesssim A$. [We try to deduce $A \sim B$ and $B \sim C$.]

  Since $B \lesssim C$ and $C \lesssim A$, we would have $B \lesssim A$.
  Then, since $A \lesssim B$ and $B \lesssim A$, we would have $A \sim B$. (Why?)

  Since $C \lesssim A$ and $A \lesssim B$, we would have $C \lesssim B$.
  Then, since $B \lesssim C$ and $C \lesssim B$, we would have $B \sim C$. (Why?)

  Hence $A \sim B$ and $B \sim C$. But by assumption, $A \not\sim B$ or $B \not\sim C$. Contradiction arises.
  Hence $A \not\sim C$ in the first place.

Then, since $A \lesssim C$ and $A \not\sim C$, we have $A < C$.

Question. *Note that* $\mathbb{Q} \lesssim \mathbb{R}$*. Is it true that* $\mathbb{Q} \sim \mathbb{R}$*, or that* $\mathbb{Q} < \mathbb{R}$*?*

Lemma (XV).

Let A, B, C *be sets. Suppose* $A \lesssim B$ *and* $B \lesssim C$*. Also suppose* $A < B$ *or* $B < C$*. Then* $A < C$*.*

Theorem (XVI).

$\mathbb{N} < [0, 1]$, *and* $\mathbb{N} < \mathbb{R}$, *and* $\mathbb{Q} < \mathbb{R}$.

Proof.

$\mathbb{N} \lesssim \mathsf{Map}(\mathbb{N}, \{0, 1\}) \lesssim \mathsf{Map}(\mathbb{N}, [\![0, 9]\!]) \sim [0, 1] \sim \mathbb{R}$.

Also, $\mathbb{N} < \mathsf{Map}(\mathbb{N}, \{0, 1\})$.

Then, by Lemma (XV), $\mathbb{N} < [0, 1]$ and $\mathbb{N} < \mathbb{R}$.

Since $\mathbb{Q} \sim \mathbb{N}$, we also have $\mathbb{Q} < \mathbb{R}$.

Remark.

Hence there are much much more real numbers than there are rational numbers.

5. **Question.** *Why are 'Venn diagram arguments' not good enough?*

Theorem (XVII.)

There exists some set T such that $S < T$ for any subset S of $\mathbb{R}^2$.

Proof.

Define $T = \mathfrak{P}(\mathbb{R})$.

Pick any subset S of $\mathbb{R}^2$. We have $S \lesssim \mathbb{R}^2 \sim \mathbb{R}$.

By Cantor's Theorem, $\mathbb{R} < \mathfrak{P}(\mathbb{R}) = T$.

Then by Lemma (XV), we have $S < T$.

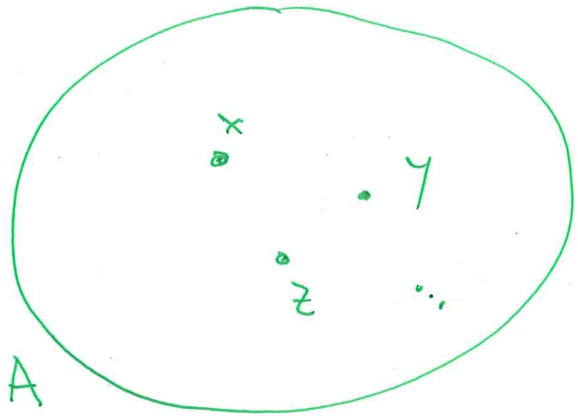
Remark.

When we draw a Venn diagram for a set, say, A, we are 'identifying' the set A with some subset, say, B, of $\mathbb{R}^2$, in the sense that the elements of A are 'identified' as the points in B, via some bijective function from A to B.

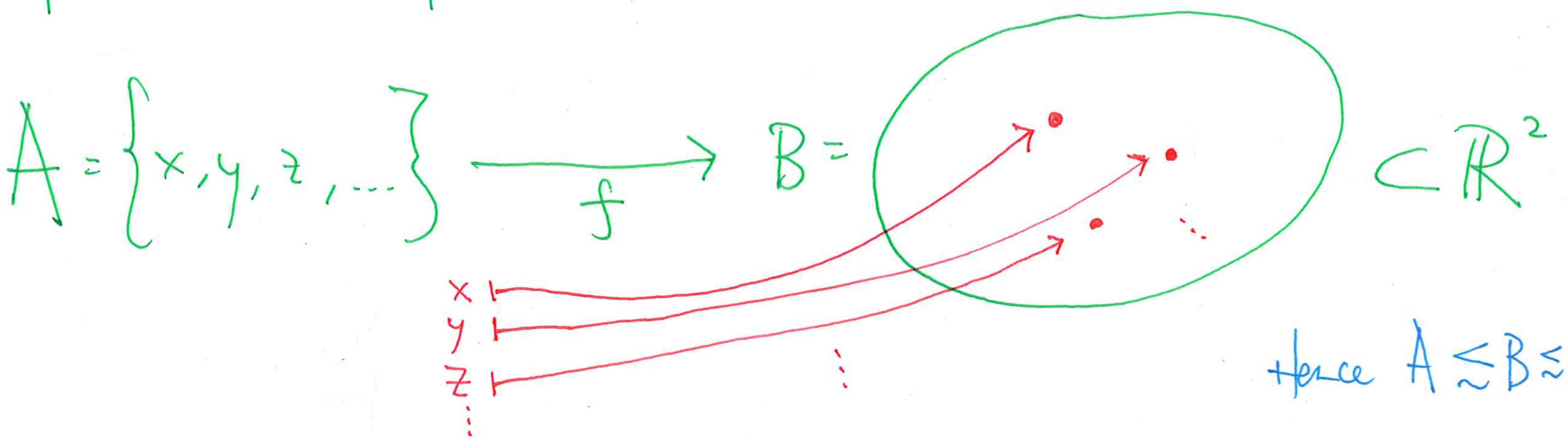
This bijective function guarantees that distinct elements of A are identified as distinct points of B. So we are implicitly assuming that there is an injective function from A to $\mathbb{R}^2$.

But now we know that there are sets which are too 'large' to be draw in a Venn diagram.

Question: What is implicit in such a 'Venn diagram' of the set $A = \{x, y, z, \dots\}$?



Answer: We are implicitly assuming that there is an injective function f from the set A to some subset B of $\mathbb{R}^2$ (on which we present the picture:



Hence $A \lesssim B \lesssim \mathbb{R}$.

6. **Question.**

Is there any 'universal set', which contains every conceivable object as its element?

Theorem (XVIII).

Denote $\{x \mid x = x\}$ by U. The mathematical object U is not a set.

Proof.

Suppose U were a set. Then, by Cantor's Theorem, $U < \mathsf{Map}(U, \{0, 1\})$.

For any $\varphi \in \mathsf{Map}(U, \{0, 1\})$, we would have $\varphi = \varphi$, and hence $\varphi \in U$.

It would follow that $\mathsf{Map}(U, \{0, 1\}) \subset U$.

Then $\mathsf{Map}(U, \{0, 1\}) \lesssim U$. Therefore $U < \mathsf{Map}(U, \{0, 1\}) \lesssim U$.

By Lemma (XV), $U < U$. In particular, $U \not\sim U$. There would be no bijective function from U to U.

But id_U is a bijective function from U to U. Contradiction arises.

Hence U is not a set in the first place.

Remark.

Hence if we insist Cantor's Theorem to be a true statement, then there is no such thing as a 'universal set'. This is known as **Cantor's Paradox**.

Cantor's Paradox

Denote $\{x \mid x = x\}$ by U.

U is not a set. (Otherwise $U \not\prec U$, or Cantor's Theorem fails to hold for U.)

Russell's Paradox

Denote $\{x \mid x \notin x\}$ by R.

R is not a set. (Otherwise '$R \in R$' is false and '$R \notin R$' is false.)