MATH1050 Cantor's diagonal argument

1. **Definition**.

   Let A, B be sets. The set $\mathsf{Map}(A, B)$ *is defined to be the* **set of all functions from** A **to** B.

   Remark. $\mathsf{Map}(\mathbb{N}, B)$ is the set of all infinite sequences in B: each $\varphi \in \mathsf{Map}(\mathbb{N}, B)$ is the infinite sequence $(\varphi(0), \varphi(1), \varphi(2), ..., \varphi(n), \varphi(n+1), ...)$, with each term being an element of B.

2. **A basic example of unequal cardinality:** $\mathbb{N} \not\sim \mathsf{Map}(\mathbb{N}, \{0,1\})$.

   Theorem (VI).

   There is no surjective function from $\mathbb{N}$ *to* $\mathsf{Map}(\mathbb{N}, \{0,1\})$.

   Corollary (VII).

   There is no bijective function from $\mathbb{N}$ *to* $\mathsf{Map}(\mathbb{N}, \{0,1\})$.

3. **Idea in the proof of Theorem (VI).**

   Suppose there were some surjective function, say, Φ, from $\mathbb{N}$ to $\mathsf{Map}(\mathbb{N}, \{0,1\})$. We look for a contradiction against this false assumption.

   For each $n \in \mathbb{N}$, the mathematical object $\Phi(n)$ is an infinite sequence in $\{0,1\}$.

   Since Φ was surjective, every infinite sequence in $\{0,1\}$ would appear somewhere in the (infinite) list of infinite sequences

$$\begin{array}{rcl} \Phi(0) &=& ((\Phi(0))(0), (\Phi(0))(1), (\Phi(0))(2), (\Phi(0))(3), \cdots), \\ \Phi(1) &=& ((\Phi(1))(0), (\Phi(1))(1), (\Phi(1))(2), (\Phi(1))(3), \cdots), \\ \Phi(2) &=& ((\Phi(2))(0), (\Phi(2))(1), (\Phi(2))(2), (\Phi(2))(3), \cdots), \\ \Phi(3) &=& ((\Phi(3))(0), (\Phi(3))(1), (\Phi(3))(2), (\Phi(3))(3), \cdots), \\ \vdots & & \vdots \end{array}$$

   Out of this list of infinite sequences, we construct an extra infinite sequence in $\{0,1\}$ which would not appear in this list. This is the desired contradiction.

   The method of construction for this extra sequence is known as **Cantor's diagonal argument**.

4. **Illustration of Cantor's diagonal argument through a 'specific' Φ.**

   For the sake of illustration, assume that $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ... are: $\Phi(0) = (0,0,1,1,1,1,\cdots)$, $\Phi(1) = (0,1,0,0,1,0,\cdots)$, $\Phi(2) = (1,0,1,1,1,0,\cdots)$, $\Phi(3) = (1,1,1,0,0,0,\cdots)$, $\Phi(4) = (0,1,1,0,0,1,\cdots)$, $\Phi(5) = (1,1,0,0,1,1,\cdots)$, ...

   List all the terms of $\Phi(0), \Phi(1), \Phi(2), \Phi(3), \Phi(4), \Phi(5), \cdots$ row by row in this 'infinite' table:

n	$\Phi(n)$	$(\Phi(n))(0)$	$(\Phi(n))(1)$	$(\Phi(n))(2)$	$(\Phi(n))(3)$	$(\Phi(n))(4)$	$(\Phi(n))(5)$	$\cdots$	diagonal	'reversi'
0	$\Phi(0)$	0	0	1	1	1	1	$\cdots$	0	1
1	$\Phi(1)$	0	1	0	0	1	0	$\cdots$	1	0
2	$\Phi(2)$	1	0	1	1	1	0	$\cdots$	1	0
3	$\Phi(3)$	1	1	1	0	0	0	$\cdots$	0	1
4	$\Phi(4)$	0	1	1	0	0	1	$\cdots$	0	1
5	$\Phi(5)$	1	1	0	0	1	1	$\cdots$	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
???	λ	1	0	0	1	1	0	$\cdots$		

   Pick out the 'diagonal' of the table and 'flip' all the 0's into 1's, and all 1's into 0's, to obtain the 'reversi' column. The 'reversi' column defines the infinite sequence λ in $\{0,1\}$: for each $n \in \mathbb{N}$, $\lambda(n)$ is the 'flip' of $(\Phi(n))(n)$.

   Since $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ... were supposedly all the infinite sequences in $\{0,1\}$, λ should be somwhere amongst $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ... Hence the row for λ should appear amongst the rows for $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ...

Question.

- *Is there any $z \in \mathbb{N}$ which satisfies $\Phi(z) = \lambda$, really?*

Answer.

- Since $\lambda(0)$ is obtained by the 'flip' of $(\Phi(0))(0)$, λ does not agree with $\Phi(0)$ at the 0-th term. Hence $\lambda \neq \Phi(0)$.
  Since $\lambda(1)$ is obtained by the 'flip' of $(\Phi(1))(1)$, λ does not agree with $\Phi(1)$ at the 1-st term. Hence $\lambda \neq \Phi(1)$.
  ...
  Since $\lambda(n)$ is obtained by the 'flip' of $(\Phi(n))(n)$, λ does not agree with $\Phi(n)$ at the n-th term. Hence $\lambda \neq \Phi(n)$.
  Et cetera.
  Hence λ is not amongst $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ... This λ is the 'extra' infinite sequence 'generated' by this 'specific' Φ according to Cantor's diagonal argument.

(No matter which 'specific' Φ we start with, the procedure described above will give you a corresponding 'troublesome' λ. Try your own favourite 'concrete' $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ..., go through the procedure described above, and have fun.)

5. **Formal argument for Theorem (VI).**

   Suppose there were some surjective function $\Phi : \mathbb{N} \longrightarrow \mathsf{Map}(\mathbb{N}, \{0, 1\})$.

   Define the function $\lambda : \mathbb{N} \longrightarrow \{0, 1\}$ by

$$\lambda(x) = \begin{cases} 1 & \text{if} \quad (\Phi(x))(x) = 0 \\ 0 & \text{if} \quad (\Phi(x))(x) = 1 \end{cases}$$

   (Is λ well-defined as a function?)

   Since Φ was surjective, there would be some $z \in \mathbb{N}$ so that $\Phi(z) = \lambda$.

   However, by definition, we have $(\Phi(z))(z) \neq \lambda(z)$. Therefore $\lambda \neq \Phi(z)$. Contradiction arises.

   Hence there is no surjective function from $\mathbb{N}$ to $\mathsf{Map}(\mathbb{N}, \{0, 1\})$ in the first place.

6. **Theorem (VIII).**

   Let A be a set. The following statements hold:

   (1) *There is no surjective function from A to $\mathsf{Map}(A, \{0, 1\})$.*

   (2) *There is no bijective function from A to $\mathsf{Map}(A, \{0, 1\})$.*

   (3) $A \not\sim \mathsf{Map}(A, \{0, 1\})$.

   Proof. The proof for Statement (1) in Theorem (VIII) is almost the same as that for Theorem (VI): just replace $\mathbb{N}$ by A in the formal proof for Theorem (VI). Statements (2), (3) follow immediately from Statement (1).

   Another formulation of Theorem (VIII).

   Let A be a set. The following statements hold:

   (1) *There is no surjective function from A to $\mathfrak{P}(A)$.*

   (2) *There is no bijective function from A to $\mathfrak{P}(A)$.*

   (3) $A \not\sim \mathfrak{P}(A)$.