1. **Definition**.

Let A, B be sets.

The set $\mathsf{Map}(A, B)$ is defined to be the **set of all functions from A to B**.

Remark.

$\mathsf{Map}(\mathbb{N}, B)$ is the set of all infinite sequences in B:

each $\varphi \in \mathsf{Map}(\mathbb{N}, B)$ is the infinite sequence

$$(\varphi(0), \varphi(1), \varphi(2), ..., \varphi(n), \varphi(n+1), ...),$$

with each term being an element of B.

2. **A basic example of unequal cardinality**: $\mathbb{N} \nsim \mathsf{Map}(\mathbb{N}, \{0, 1\})$.

Theorem (VI).

There is no surjective function from $\mathbb{N}$ to $\mathsf{Map}(\mathbb{N}, \{0, 1\})$.

Corollary (VII).

There is no bijective function from $\mathbb{N}$ to $\mathsf{Map}(\mathbb{N}, \{0, 1\})$.

3. **Idea in the proof of Theorem (VI).**

Suppose there were some surjective function, say, Φ, from N to $\mathsf{Map}(\mathsf{N}, \{0,1\})$.

We look for a contradiction against this false assumption.

For each $n \in \mathsf{N}$, the mathematical object $\Phi(n)$ is an infinite sequence in $\{0,1\}$.

Since Φ was surjective, *every* infinite sequence in $\{0,1\}$ would *appear somewhere* in the (infinite) list of infinite sequences

$$\begin{array}{rcl}
\Phi(0) & = & ((\Phi(0))(0), (\Phi(0))(1), (\Phi(0))(2), (\Phi(0))(3), \cdots), \\
\Phi(1) & = & ((\Phi(1))(0), (\Phi(1))(1), (\Phi(1))(2), (\Phi(1))(3), \cdots), \\
\Phi(2) & = & ((\Phi(2))(0), (\Phi(2))(1), (\Phi(2))(2), (\Phi(2))(3), \cdots), \\
\Phi(3) & = & ((\Phi(3))(0), (\Phi(3))(1), (\Phi(3))(2), (\Phi(3))(3), \cdots), \\
\vdots & & \vdots
\end{array}$$

Out of this list of infinite sequences, we construct an *extra* infinite sequence in $\{0,1\}$ which would *not appear* in this list. This is the desired contradiction.

The method of construction for this extra sequence is known as **Cantor's diagonal argument**.

4. **Illustration of Cantor's diagonal argument through a 'specific' Φ.**

Here $\Phi : \mathbb{N} \longrightarrow \mathsf{Map}(\mathbb{N}, \{0,1\})$ is supposed to be surjective.

For the sake of illustration, assume that $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ... are:

$$
\begin{array}{l|l|l}
\Phi(0) = (0,0,1,1,1,1,\cdots), & \Phi(3) = (1,1,1,0,0,0,\cdots), & \\
\Phi(1) = (0,1,0,0,1,0,\cdots), & \Phi(4) = (0,1,1,0,0,1,\cdots), & \\
\Phi(2) = (1,0,1,1,1,0,\cdots), & \Phi(5) = (1,1,0,0,1,1,\cdots), & \cdots\cdots\cdots
\end{array}
$$

List all the terms of $\Phi(0), \Phi(1), \Phi(2), \Phi(3), \Phi(4), \Phi(5), \cdots$ row by row in this 'infinite' table:

n	$\Phi(n)$	$(\Phi(n))(0)$	$(\Phi(n))(1)$	$(\Phi(n))(2)$	$(\Phi(n))(3)$	$(\Phi(n))(4)$	$(\Phi(n))(5)$	$\cdots$		
0	$\Phi(0)$							$\cdots$		
1	$\Phi(1)$							$\cdots$		
2	$\Phi(2)$							$\cdots$		
3	$\Phi(3)$							$\cdots$		
4	$\Phi(4)$							$\cdots$		
5	$\Phi(5)$							$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
???								$\cdots$		

What do we expect for this table by virture of the surjectivity of Φ?

Illustration of Cantor's diagonal argument through a 'specific' Φ.

Here $\Phi : \mathbb{N} \longrightarrow \mathsf{Map}(\mathbb{N}, \{0, 1\})$ is supposed to be surjective.

For the sake of illustration, assume that $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ... are:

$$
\begin{array}{l|l|l}
\Phi(0) = (0, 0, 1, 1, 1, 1, \cdots), & \Phi(3) = (1, 1, 1, 0, 0, 0, \cdots), & \\
\Phi(1) = (0, 1, 0, 0, 1, 0, \cdots), & \Phi(4) = (0, 1, 1, 0, 0, 1, \cdots), & \\
\Phi(2) = (1, 0, 1, 1, 1, 0, \cdots), & \Phi(5) = (1, 1, 0, 0, 1, 1, \cdots), & \cdots\cdots\cdots
\end{array}
$$

List all the terms of $\Phi(0), \Phi(1), \Phi(2), \Phi(3), \Phi(4), \Phi(5), \cdots$ row by row in this 'infinite' table:

n	$\Phi(n)$	$(\Phi(n))(0)$	$(\Phi(n))(1)$	$(\Phi(n))(2)$	$(\Phi(n))(3)$	$(\Phi(n))(4)$	$(\Phi(n))(5)$	$\cdots$		
0	$\Phi(0)$	0	0	1	1	1	1	$\cdots$		
1	$\Phi(1)$	0	1	0	0	1	0	$\cdots$		
2	$\Phi(2)$	1	0	1	1	1	0	$\cdots$		
3	$\Phi(3)$	1	1	1	0	0	0	$\cdots$		
4	$\Phi(4)$	0	1	1	0	0	1	$\cdots$		
5	$\Phi(5)$	1	1	0	0	1	1	$\cdots$		
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
???								$\cdots$		

Because Φ is surjective, we expect every infinite binary sequence turns up somewhere amongst the rows of this table.

Illustration of Cantor's diagonal argument through a 'specific' Φ.

Here $\Phi : \mathbb{N} \longrightarrow \mathsf{Map}(\mathbb{N}, \{0,1\})$ is supposed to be surjective.

For the sake of illustration, assume that $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ... are:

$$\Phi(0) = (0,0,1,1,1,1,\cdots), \qquad \Phi(3) = (1,1,1,0,0,0,\cdots),$$
$$\Phi(1) = (0,1,0,0,1,0,\cdots), \qquad \Phi(4) = (0,1,1,0,0,1,\cdots),$$
$$\Phi(2) = (1,0,1,1,1,0,\cdots), \qquad \Phi(5) = (1,1,0,0,1,1,\cdots), \qquad \cdots\cdots\cdots$$

List all the terms of $\Phi(0), \Phi(1), \Phi(2), \Phi(3), \Phi(4), \Phi(5), \cdots$ row by row in this 'infinite' table:

n	$\Phi(n)$	$(\Phi(n))(0)$	$(\Phi(n))(1)$	$(\Phi(n))(2)$	$(\Phi(n))(3)$	$(\Phi(n))(4)$	$(\Phi(n))(5)$	$\cdots$	diagonal	'reversi'
0	$\Phi(0)$	0	0	1	1	1	1	$\cdots$	0	1
1	$\Phi(1)$	0	1	0	0	1	0	$\cdots$	1	0
2	$\Phi(2)$	1	0	1	1	1	0	$\cdots$	1	0
3	$\Phi(3)$	1	1	1	0	0	0	$\cdots$	0	1
4	$\Phi(4)$	0	1	1	0	0	1	$\cdots$	0	1
5	$\Phi(5)$	1	1	0	0	1	1	$\cdots$	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
???								$\cdots$		

'Flip' the respective entries in the 'diagonal column'.

Study the diagonal of this table.

Because Φ is surjective, we expect every infinite binary sequence turns up somewhere amongst the rows of this table.

Illustration of Cantor's diagonal argument through a 'specific' Φ.

Here $\Phi : \mathbb{N} \longrightarrow \mathsf{Map}(\mathbb{N}, \{0,1\})$ is supposed to be surjective.

For the sake of illustration, assume that $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ... are:

$$\Phi(0) = (0,0,1,1,1,1,\cdots), \quad \Phi(3) = (1,1,1,0,0,0,\cdots),$$
$$\Phi(1) = (0,1,0,0,1,0,\cdots), \quad \Phi(4) = (0,1,1,0,0,1,\cdots),$$
$$\Phi(2) = (1,0,1,1,1,0,\cdots), \quad \Phi(5) = (1,1,0,0,1,1,\cdots), \quad \cdots\cdots\cdots$$

List all the terms of $\Phi(0), \Phi(1), \Phi(2), \Phi(3), \Phi(4), \Phi(5), \cdots$ row by row in this 'infinite' table:

n	$\Phi(n)$	$(\Phi(n))(0)$	$(\Phi(n))(1)$	$(\Phi(n))(2)$	$(\Phi(n))(3)$	$(\Phi(n))(4)$	$(\Phi(n))(5)$	$\cdots$	diagonal	'reversi'
0	$\Phi(0)$	0	0	1	1	1	1	$\cdots$	0	1
1	$\Phi(1)$	0	1	0	0	1	0	$\cdots$	1	0
2	$\Phi(2)$	1	0	1	1	1	0	$\cdots$	1	0
3	$\Phi(3)$	1	1	1	0	0	0	$\cdots$	0	1
4	$\Phi(4)$	0	1	1	0	0	1	$\cdots$	0	1
5	$\Phi(5)$	1	1	0	0	1	1	$\cdots$	1	0
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$	$\vdots$
???	λ	1	0	0	1	1	0	$\cdots$		
		$\lambda(0)$	$\lambda(1)$	$\lambda(2)$	$\lambda(3)$	$\lambda(4)$	$\lambda(5)$	$\cdots$		

'Transpose' of the 'reversi column'.

λ is certainly an infinite binary sequence.

Because Φ is surjective, we expect every infinite binary sequence turns up somewhere amongst the rows of this table. **But how about λ? Nowhere. Why?!**

Question.

- *Is there any $z \in \mathbb{N}$ which satisfies $\Phi(z) = \lambda$, really?*

Answer.

- Since $\lambda(0)$ is obtained by the 'flip' of $(\Phi(0))(0)$, λ does not agree with $\Phi(0)$ at the 0-th term. Hence $\lambda \neq \Phi(0)$.

  Since $\lambda(1)$ is obtained by the 'flip' of $(\Phi(1))(1)$, λ does not agree with $\Phi(1)$ at the 1-st term. Hence $\lambda \neq \Phi(1)$. ...

  Since $\lambda(n)$ is obtained by the 'flip' of $(\Phi(n))(n)$, λ does not agree with $\Phi(n)$ at the n-th term. Hence $\lambda \neq \Phi(n)$. Et cetera.

  Hence λ is not amongst $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ...

  This λ is the '*extra*' infinite sequence 'generated' by this 'specific' Φ according to Cantor's diagonal argument.

(No matter which 'specific' Φ we start with, the procedure described above will give you a corresponding 'troublesome' λ. Try your own favourite 'concrete' $\Phi(0)$, $\Phi(1)$, $\Phi(2)$, $\Phi(3)$, $\Phi(4)$, $\Phi(5)$, ..., go through the procedure described above, and have fun.)

5. **Formal argument for Theorem (VI).**

Suppose there were some surjective function $\Phi : \mathsf{N} \longrightarrow \mathsf{Map}(\mathsf{N}, \{0, 1\})$.

Define the function $\lambda : \mathsf{N} \longrightarrow \{0, 1\}$ by

$$\lambda(x) = \begin{cases} 1 & \text{if} \quad (\Phi(x))(x) = 0 \\ 0 & \text{if} \quad (\Phi(x))(x) = 1 \end{cases}$$

(Is λ well-defined as a function?)

Since Φ was surjective, there would be some $z \in \mathsf{N}$ so that $\Phi(z) = \lambda$.

However, by definition, we have $(\Phi(z))(z) \neq \lambda(z)$.

Therefore $\lambda \neq \Phi(z)$. Contradiction arises.

Hence there is no surjective function from N to $\mathsf{Map}(\mathsf{N}, \{0, 1\})$ in the first place.

6. **Theorem (VIII)**.

Let A be a set. The following statements hold:

(1) *There is no surjective function from A to* $\mathsf{Map}(A, \{0,1\})$.

(2) *There is no bijective function from A to* $\mathsf{Map}(A, \{0,1\})$.

(3) $A \not\sim \mathsf{Map}(A, \{0,1\})$.

Proof.

The proof for Statement (1) in Theorem (VIII) is almost the same as that for Theorem (VI): just replace $\mathbb{N}$ by A in the formal proof for Theorem (VI).

Statements (2), (3) follow immediately from Statement (1).

Proof of Statement (1) in Theorem $(\boxed{\text{VIII}})$.

~~Formal argument for Theorem (VI)~~.

Suppose there were some surjective function $\Phi : \cancel{\mathbb{N}}\,A \longrightarrow \mathsf{Map}(\cancel{\mathbb{N}}\,A, \{0, 1\})$.

Define the function $\lambda : \cancel{\mathbb{N}}\,A \longrightarrow \{0, 1\}$ by

$$\lambda(x) = \begin{cases} 1 & \text{if} \quad (\Phi(x))(x) = 0 \\ 0 & \text{if} \quad (\Phi(x))(x) = 1 \end{cases}$$

(Is λ well-defined as a function?)

Since Φ was surjective, there would be some $z \in \cancel{\mathbb{N}}\,A$ so that $\Phi(z) = \lambda$.

However, by definition, we have $(\Phi(z))(z) \neq \lambda(z)$.

Therefore $\lambda \neq \Phi(z)$. Contradiction arises.

Hence there is no surjective function from $\cancel{\mathbb{N}}\,A$ to $\mathsf{Map}(\cancel{\mathbb{N}}\,A, \{0, 1\})$ in the first place.

Theorem (VIII).

Let A be a set. The following statements hold:

(1) *There is no surjective function from A to* $\mathsf{Map}(A, \{0, 1\})$.

(2) *There is no bijective function from A to* $\mathsf{Map}(A, \{0, 1\})$.

(3) $A \not\sim \mathsf{Map}(A, \{0, 1\})$.

Another formulation of Theorem (VIII).

Let A be a set. The following statements hold:

(1) *There is no surjective function from A to* $\mathfrak{P}(A)$.

(2) *There is no bijective function from A to* $\mathfrak{P}(A)$.

(3) $A \not\sim \mathfrak{P}(A)$.

Why? How?

- Recall that $\mathfrak{P}(A) \sim \mathrm{Map}(A, \{0,1\})$.

  $\chi : \mathfrak{P}(A) \longrightarrow \mathrm{Map}(A, \{0,1\})$ given by

  $\chi^A(S) = \chi_S^A$ for any $S \in \mathfrak{P}(A)$

  is a bijective function.