MATH1050 Sets of equal cardinality

1. **Definition.**
   Let A, B be sets. We say that A is **of cardinality equal to** B *if there is a bijective function from A to B. We write $A \sim B$.*

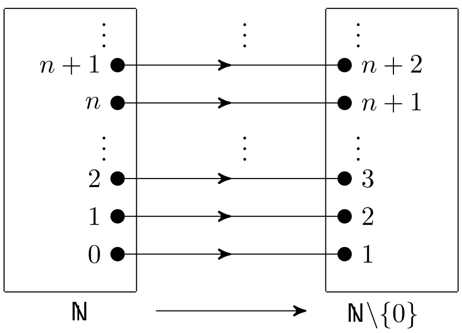
   Remark on notation. Where A is not of cardinality equal to B, we write $A \not\sim B$.

2. **Theorem (I). (Properties of $\sim$.)**
   (1) *Suppose A is a set. Then $A \sim \emptyset$ iff $A = \emptyset$.*
   (2) *Suppose x, y are objects. Then $\{x\} \sim \{y\}$.*
   (3) *Let A, B, C be sets. The following statements hold:*
      (3a) *$A \sim A$.*
      (3b) *Suppose $A \sim B$. Then $B \sim A$.*
      (3c) *Suppose $A \sim B$ and $B \sim C$. Then $A \sim C$.*
   (4) *Let A, B, C, D be sets. The following statements hold:*
      (4a) *Suppose $A \sim C$ and $B \sim D$. Then $A \times B \sim C \times D$.*
      (4b) *Suppose $A \sim C$. Then $\mathfrak{P}(A) \sim \mathfrak{P}(C)$.*
      (4c) *Suppose $A \sim C$ and $B \sim D$. Then $\mathsf{Map}(A, B) \sim \mathsf{Map}(C, D)$.*

   Remarks.
   - According to (3), $\sim$ defines an equivalence relation in the power set of any given set.
   - In (4), $\mathsf{Map}(A, B)$ is the set of all functions from A to B.

3. **Example (α).**
   $\mathbb{N} \sim \mathbb{N} \backslash \{0\}$.

   (a) *Idea.*



   This is the 'blobs-and-arrows' diagram for a certain bijective function, which we denote by f here, but how to write down this f explicitly?
   It is the function $f : \mathbb{N} \longrightarrow \mathbb{N} \backslash \{0\}$ whose graph is $\{(x, x+1) \mid x \in \mathbb{N}\}$ respectively.
   Its 'formula of definition' is given by $f(x) = x + 1$ for any $x \in \mathbb{N}$.

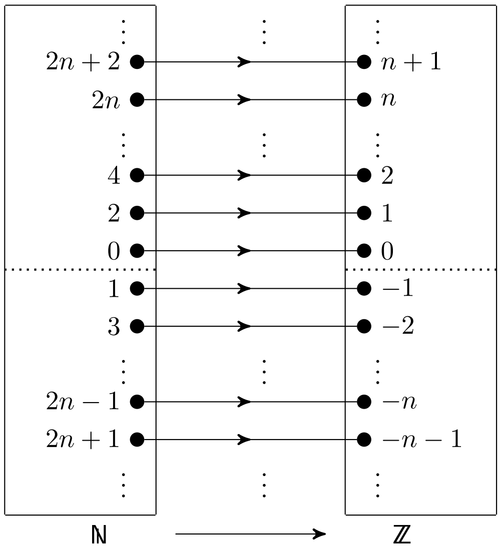
   (b) *Formal argument.*
   Let $F = \{(x, x+1) \mid x \in \mathbb{N}\}$.
   (Very formally presented, we have $F = \{p \mid$ There exists some $x \in \mathbb{N}$ such that $p = (x, x+1).\}$.)
   Note that $F \subset \mathbb{N} \times (\mathbb{N} \backslash \{0\})$.
   Define $f = (\mathbb{N}, \mathbb{N} \backslash \{0\}, F)$.
   f is a relation from $\mathbb{N}$ to $\mathbb{N} \backslash \{0\}$.
   Now we proceed to verify that f is a bijective function:
   * Pick any $x \in \mathbb{N}$. Take $y = x + 1$. Since $x, 1 \in \mathbb{N}$, we have $y \in \mathbb{N}$. Moreover, $y = x + 1 \geq 0 + 1 > 0$. Then $y \in \mathbb{N} \backslash \{0\}$. By definition, $(x, y) \in F$.
   * Pick any $x \in \mathbb{N}$. Pick any $y, z \in \mathbb{N} \backslash \{0\}$. Suppose $(x, y) \in F$ and $(x, z) \in F$. Since $(x, y) \in F$, there exists some $u \in \mathbb{N}$ such that $(x, y) = (u, u+1)$. Since $(x, z) \in F$, there exists some $v \in \mathbb{N}$ such that $(x, z) = (v, v+1)$. Now we have $u = x = v$. Then $y = u + 1 = v + 1 = z$.

- ∗ Hence $f : \mathbb{N} \longrightarrow \mathbb{N}\backslash\{0\}$ is indeed a function, given by $f(x) = x + 1$ for any $x \in \mathbb{N}$.
- ∗ Pick any $y \in \mathbb{N}\backslash\{0\}$. Take $x = y - 1$. Since $y, 1 \in \mathbb{Z}$, we have $x \in \mathbb{Z}$. Since $y \geq 1$, we have $x = y - 1 \geq 0$. Then $x \in \mathbb{N}$. By definition, $f(x) = x + 1 = (y - 1) + 1 = y$.
- ∗ Pick any $w, x \in \mathbb{N}$. Suppose $f(x) = f(w)$. Then $x - 1 = w - 1$. Therefore $w = x$.
- ∗ It follows that f is a bijective function from $\mathbb{N}$ to $\mathbb{N}\backslash\{0\}$.

4. **Example (β).**

   $\mathbb{N} \sim \mathbb{Z}$.

   (a) *Idea.*



   (b) *Formal argument.*

   Let $F_1 = \{(2x, x) \mid x \in \mathbb{N}\}$, $F_2 = \{(2x - 1, -x) \mid x \in \mathbb{N}\backslash\{0\}\}$, and $F = F_1 \cup F_2$.

   Note that $F \subset \mathbb{N} \times \mathbb{Z}$.

   Define $f = (\mathbb{N}, \mathbb{Z}, F)$. f is a relation from $\mathbb{N}$ to $\mathbb{Z}$.

   Now verify that f is a bijective function. (Fill in the details. Theorem (II) may help.)

   The 'formula of definition' of the bijective function $f : \mathbb{N} \longrightarrow \mathbb{Z}$ is given by

$$f(x) = \begin{cases} \dfrac{x}{2} & \text{if } x \text{ is even} \\ -\dfrac{x+1}{2} & \text{if } x \text{ is odd} \end{cases}$$

5. **'Glueing Lemma'.**

   Theorem (II). ('Baby version' of 'Glueing Lemma').

   Let C, C', D, D' be sets, and $g = (C, D, G), g' = (C', D', G')$ be bijective functions. Suppose $C \cap C' = \emptyset$ and $D \cap D' = \emptyset$. Then $(C \cup C', D \cup D', G \cup G')$ is a bijective function.

   Corollary (III).

   Let C, C', D, D' be sets. Suppose $C \sim D$ and $C' \sim D'$. Also suppose $C \cap C' = \emptyset$ and $D \cap D' = \emptyset$. Then $C \cup C' \sim D \cup D'$.

   Theorem (II) and Corollary (III) may be extended to the situation for infinite sequences of sets and generalized unions:

   Theorem (IV). ('Glueing Lemma'.)

   Let A, B be sets. Let $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$ be infinite sequences of subsets of A, B respectively. Let $\{G_n\}_{n=0}^{\infty}$ be an infinite seuqence of subsets of $A \times B$. Suppose $\{(C_n, D_n, G_n)\}_{n=0}^{\infty}$ is an infinite sequence of bijective functions. Suppose that for any $j, k \in \mathbb{N}$, if $j \neq k$ then $C_j \cap C_k = \emptyset$ and $D_j \cap D_k = \emptyset$. Then $\left(\bigcup_{n=0}^{\infty} C_n, \bigcup_{n=0}^{\infty} D_n, \bigcup_{n=0}^{\infty} G_n\right)$ is a bijective function.

   Corollary (V).

   Let A, B be sets. Let $\{C_n\}_{n=0}^{\infty}$, $\{D_n\}_{n=0}^{\infty}$ be infinite sequences of subsets of A, B respectively. Suppose that for any $n \in \mathbb{N}$, $C_n \sim D_n$. Also suppose that for any $j, k \in \mathbb{N}$, if $j \neq k$ then $C_j \cap C_k = \emptyset$ and $D_j \cap D_k = \emptyset$. Then $\bigcup_{n=0}^{\infty} C_n \sim \bigcup_{n=0}^{\infty} D_n$.

6. **Example** (γ).

$\mathbb{N} \sim \mathbb{N}^2$.

Remark. Hence, by Theorem (I) and the result in Example (β), we have $\mathbb{N}^m \sim \mathbb{N}$ and $\mathbb{Z}^m \sim \mathbb{Z}$ for any $m \in \mathbb{N}^*$.

(a) *Idea.*

Break up each of $\mathbb{N}$, $\mathbb{N}^2$ into many many parts, match the parts with bijective functions, and then 'glue up' these bijective functions to obtain a bijective function from $\mathbb{N}$ to $\mathbb{N}^2$.

There are many ways to do it.

(b) *Correspondence 1.*

$$\begin{array}{c|ccc|ccccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\
(0,0) & (1,0) & (1,1) & (0,1) & (2,0) & (2,1) & (2,2) & (1,2) & (0,2) & \ldots
\end{array}$$

We have constructed the bijective function $f_1 : \mathbb{N} \longrightarrow \mathbb{N}^2$ below which 'matches' the respective entries at the corresponding positions of the following 'infinite square-arrays' to each other:

$$\left|\begin{array}{ccccccc}
0 & 1 & 4 & 9 & 16 & 25 & \ldots \\
3 & 2 & 5 & 10 & 17 & 26 & \ldots \\
8 & 7 & 6 & 11 & 18 & 27 & \ldots \\
15 & 14 & 13 & 12 & 19 & 28 & \ldots \\
24 & 23 & 22 & 21 & 20 & 29 & \ldots \\
35 & 34 & 33 & 32 & 31 & 30 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right| \longrightarrow \left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \ldots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \ldots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \ldots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \ldots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \ldots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|$$

(c) *Correspondence 2.*

$$\begin{array}{c|cc|ccc|cccc|c}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & \ldots \\
\downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\
(0,0) & (1,0) & (0,1) & (2,0) & (1,1) & (0,2) & (3,0) & (2,1) & (1,2) & (0,3) & \ldots
\end{array}$$

We have constructed the bijective function $f_2 : \mathbb{N} \longrightarrow \mathbb{N}^2$ below which 'matches' the respective entries at the corresponding positions of the following 'infinite square-arrays' to each other:

$$\left|\begin{array}{ccccccc}
0 & 1 & 3 & 6 & 10 & 15 & \ldots \\
2 & 4 & 7 & 11 & 16 & 22 & \ldots \\
5 & 8 & 12 & 17 & 23 & 30 & \ldots \\
9 & 13 & 18 & 24 & 31 & 39 & \ldots \\
14 & 19 & 25 & 32 & 40 & 49 & \ldots \\
20 & 26 & 33 & 41 & 50 & 60 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right| \longrightarrow \left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \ldots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \ldots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \ldots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \ldots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \ldots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|$$

(d) *Correspondence 3.*

Define $g : \mathbb{N}^2 \longrightarrow \mathbb{N}\backslash\{0\}$ by $g(x, y) = 2^y(2x+1)$ for any $x, y \in \mathbb{N}$. g is a bijective function. g sets up the following 'exact correspondence' from $\mathbb{N}^2$ to $\mathbb{N}\backslash\{0\}$:

$$\left|\begin{array}{ccccccc}
(0,0) & (1,0) & (2,0) & (3,0) & (4,0) & (5,0) & \ldots \\
(0,1) & (1,1) & (2,1) & (3,1) & (4,1) & (5,1) & \ldots \\
(0,2) & (1,2) & (2,2) & (3,2) & (4,2) & (5,2) & \ldots \\
(0,3) & (1,3) & (2,3) & (3,3) & (4,3) & (5,3) & \ldots \\
(0,4) & (1,4) & (2,4) & (3,4) & (4,4) & (5,4) & \ldots \\
(0,5) & (1,5) & (2,5) & (3,5) & (4,5) & (5,5) & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right| \longrightarrow \left|\begin{array}{ccccccc}
1 & 3 & 5 & 7 & 9 & 11 & \ldots \\
2 & 6 & 10 & 14 & 18 & 22 & \ldots \\
4 & 12 & 20 & 28 & 36 & 44 & \ldots \\
8 & 24 & 40 & 56 & 72 & 88 & \ldots \\
16 & 48 & 80 & 112 & 144 & 176 & \ldots \\
32 & 96 & 160 & 224 & 288 & 352 & \ldots \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots
\end{array}\right|$$

Define $h : \mathbb{N}\backslash\{0\} \longrightarrow \mathbb{N}$ by $h(w) = w - 1$ for any $w \in \mathbb{N}\backslash\{0\}$. h is a bijective function. Now $h \circ g$ is a bijective function from $\mathbb{N}^2$ to $\mathbb{N}$, given by $(h \circ g)(x, y) = 2^y(2x + 1) - 1$ for any $x, y \in \mathbb{N}$.

7. **Example** (δ).

Suppose I is an interval with more than one point. Then $I \sim \mathbb{R}$.

- *Outline of argument*:

  (a) Suppose I is 'finite at both ends'. Deduce:

  (a1) $I \sim [0, 1]$ if I is closed.

(a2) $I \sim [0,1)$ if I is half-closed-half-open.

(a3) $I \sim (0,1)$ if I is open.

(b) Suppose $I \neq \mathbb{R}$ and I is not 'finite at both ends'. Deduce:

(b1) $I \sim [0,+\infty)$ if I is closed.

(b2) $I \sim (0,+\infty)$ if I is open.

(c) Deduce that $[0,1] \sim [0,1)$. Similarly deduce that $[0,1) \sim (0,1)$.

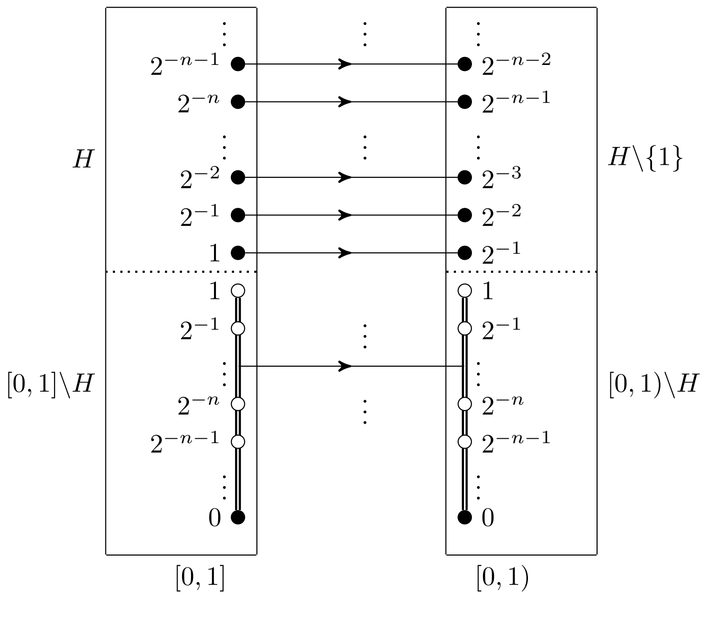
(d) Deduce that $(0,1) \sim (0,+\infty)$. Similarly deduce that $[0,1) \sim [0,+\infty)$.

(e) Deduce that $(0,1) \sim \mathbb{R}$.

- Respective arguments for (a), (b): Make use of 'linear functions'.
  Respective arguments for (d), (e): Make use of 'rational functions'.
  Argument for (c)? This is non-trivial.

Argument for (c):

- *Idea.*
  $[0,1)$ is almost the whole of $[0,1]$ except that it 'misses' the point 1. Try to 'modify' the identity function from $[0,1]$ to $[0,1]$ to get a bijective function from $[0,1]$ to $[0,1)$.
- *Trick.*
  Dig many many holes in $[0,1]$, $[0,1)$ at identical positions so that after this digging, what remain of these two sets are the same set.
  (But what to do with the 'debris'? Don't throw them away.)
  Take $H = \left\{ \dfrac{1}{2^n} \;\middle|\; n \in \mathbb{N} \right\}$. It is the set of all terms of the strictly decreasing infinite sequence $\left\{ \dfrac{1}{2^n} \right\}_{n=0}^{\infty}$ in $[0,1]$.
  Except its zero-th term, every term is in $[0,1)$.
  Now draw the 'blobs-and-arrows diagram' as described here:
  * Match 1 in $[0,1]$ with $\dfrac{1}{2}$ in $[0,1)$. Match $\dfrac{1}{2}$ in $[0,1]$ with $\dfrac{1}{4}$ in $[0,1)$. Match $\dfrac{1}{4}$ in $[0,1]$ with $\dfrac{1}{8}$ in $[0,1)$. ...
    Match $\dfrac{1}{2^n}$ in $[0,1]$ with $\dfrac{1}{2^{n+1}}$ in $[0,1)$. Match $\dfrac{1}{2^{n+1}}$ in $[0,1]$ with $\dfrac{1}{2^{n+2}}$ in $[0,1)$. Et cetera.
  * Now note that $[0,1]\backslash H = [0,1)\backslash H$. So we match these two sets with the identity function.



- *Formal argument.*
  Define $H = \left\{ \dfrac{1}{2^n} \;\middle|\; n \in \mathbb{N} \right\}$. Note that $[0,1]\backslash H = [0,1)\backslash H$.
  Define $F_1 = \{(x,x) \mid x \in [0,1]\backslash H\}$ and $F_2 = \left\{ (x, \dfrac{x}{2}) \mid x \in H \right\}$ and $F = F_1 \cup F_2$.
  Verify that $f_1 = ([0,1]\backslash H, [0,1)\backslash H, F_1)$, $f_2 = (H, H\backslash\{1\}, F_2)$ are bijective functions. (Fill in the detail.)
  Define $f = ([0,1],[0,1),F)$. f is a relation. f is a bijective function according to the 'Glueing Lemma'.
- The argument for $[0,1) \sim (0,1)$ is similar.

8. **Example** (ϵ)**.**

Suppose A is a set. Then $\mathfrak{P}(A) \sim \mathsf{Map}(A, \{0,1\})$.

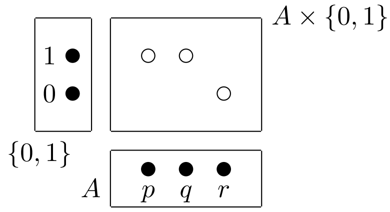
Remark. $\mathsf{Map}(A, \{0,1\})$ is the set of all functions from A to $\{0,1\}$.

(a) *Idea* (through one example).

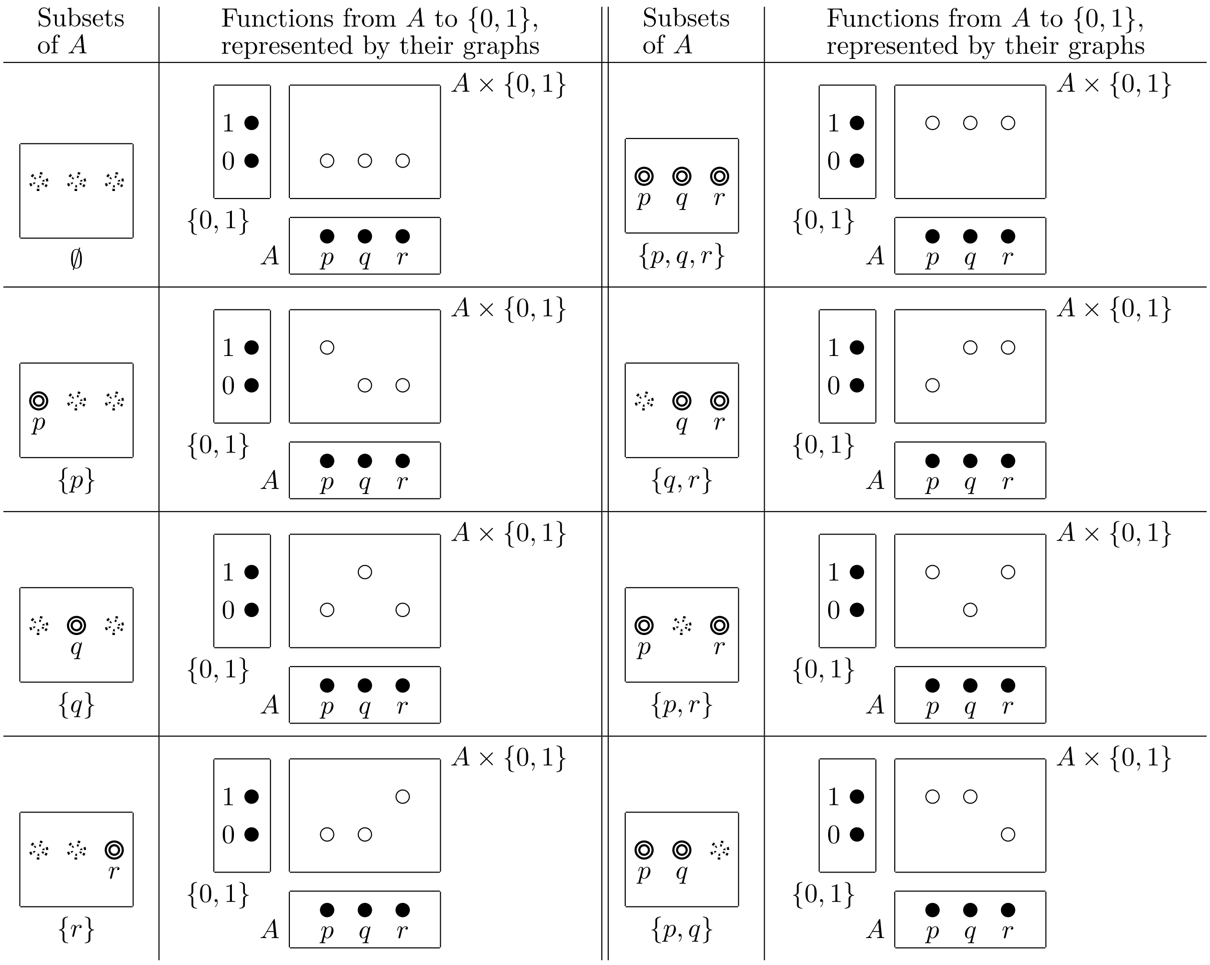
Let $A = \{p, q, r\}$, where p, q, r are pairwise distinct.

'Light bulb' analogy:

- Imagine p, q, r are points on the plane, and a light bulb is fixed at each of p, q, r.
- When a subset S of A is named, we turn on the lights at the corresponding elements of S.
  The light-bulbs at the elements of S go to 'on-state' (denoted by '1').
  The 'light-bulbs' at the elements of $A \backslash S$ remain in the 'off-state' (denoted by '0').
  This give an 'overall state' of the 'light bulbs' in A according to what S is.
- For instance, when $S = \{p, q\}$, the lightbulbs at p, q are 'on' and that at r remains 'off'. We may represent this overall state in such a diagram:



- Such a diagram is in fact a graph of the function from A to $\{0,1\}$.
  (When $S = \{p, q\}$, the function concerned assigns p, q, r to $1, 1, 0$ respectively.)
- *Observation.*
  Each individual element of $\mathfrak{P}(A)$ corresponds to exactly one 'overall state' of the "light-bulbs" in A.
  So we have a 'natural' 'exact correspondence' between the subsets of A and the functions from A to $\{0,1\}$ (as visualized by their respective graphs).



(b) *Formal argument.*

Suppose A is a set. Then $A = \emptyset$ or $A \neq \emptyset$.

If $A = \emptyset$ then $(\mathfrak{P}(A) = \{\emptyset\}$ and $\mathsf{Map}(A, \{0,1\}) = \{(\emptyset, \{0,1\}, \emptyset)\})$. [Done.]

From now on suppose $A \neq \emptyset$. For each $S \in \mathfrak{P}(A)$, define the function $\chi_S^A : A \longrightarrow \{0,1\}$ by

$$\chi_S^A(x) = \begin{cases} 1 & \text{if} \quad x \in S \\ 0 & \text{if} \quad x \in A \backslash S. \end{cases}$$

Define the function $f : \mathfrak{P}(A) \longrightarrow \mathsf{Map}(A, \{0,1\})$ by $f(S) = \chi_S^A$ for any $S \in \mathfrak{P}(A)$.

Verify that f is bijective. (Fill in the detail.)

Remark. χ_S^A is called the **characteristic function of the set S in the set A.**

9. **Example (ζ).**

$\mathsf{Map}(\mathbb{N}, \{0,1\}) \sim (\mathsf{Map}(\mathbb{N}, \{0,1\}))^2$.

Remark. $\mathsf{Map}(\mathbb{N}, \{0,1\})$ is the set of all functions from $\mathbb{N}$ to $\{0,1\}$.

- Each function $\varphi : \mathbb{N} \longrightarrow \{0,1\}$ is uniquely identified as the binary infinite sequence $(\varphi(0), \varphi(1), \varphi(2), \varphi(3), \cdots)$.
- Each binary infinite sequence $(a_0, a_1, a_2, a_3, \cdots)$ is uniquely identified as the function $a : \mathbb{N} \longrightarrow \{0,1\}$ given by $a(k) = a_k$ for any $k \in \mathbb{N}$.

Hence $\mathsf{Map}(\mathbb{N}, \{0,1\})$ is the set of all binary infinite sequences.

(a) *Idea.*

Each element of $\mathsf{Map}(\mathbb{N}, \{0,1\})$ is a function from $\mathbb{N}$ to $\{0,1\}$, and hence is an infinite sequence in $\{0,1\}$.

Is there any natural 'exact correspondence' between infinite sequences in $\{0,1\}$ and ordered pairs of such sequences?

- ∗ Just name any infinite sequence in $\{0,1\}$. For convenicence, call it $\{a_n\}_{n=0}^{\infty}$.
- ∗ What do we obtain from $\{a_n\}_{n=0}^{\infty}$ by deleting all terms at 'odd positions?', without changing the ordering of the terms?
- ∗ What do we obtain from $\{a_n\}_{n=0}^{\infty}$ by deleting all terms at 'even positions?', without changing the ordering of the terms?
- ∗ Can we recover the original infinite sequence $\{a_n\}_{n=0}^{\infty}$ from the two resultant infinite sequences?

What can we say about the function from $\mathsf{Map}(\mathbb{N}, \{0,1\})$ to $(\mathsf{Map}(\mathbb{N}, \{0,1\}))^2$ defined by

$$(a_0, a_1, a_2, a_3, a_4, a_5, \cdots) \longmapsto ((a_0, a_2, a_4, \cdots), (a_1, a_3, a_5, \cdots))$$

for each infinite sequence $\{a_n\}_{n=0}^{\infty}$ in $\{0,1\}$?

(b) *Formal argument.*

Exercise.

Remarks. More generally, we have:

(a) $\mathsf{Map}(\mathbb{N}, \{0,1\}) \sim (\mathsf{Map}(\mathbb{N}, \{0,1\}))^n$ for any $n \in \mathbb{N} \backslash \{0\}$.

(b) $\mathsf{Map}(\mathbb{N}, B) \sim (\mathsf{Map}(\mathbb{N}, B))^n$ for any $n \in \mathbb{N} \backslash \{0\}$, whenever B is a non-empty set.