MATH1050 Examples: Relations.

1. Define the relation $T = (\mathbb{R}, \mathbb{R}, G)$ in $\mathbb{R}$ by $G = \{(x, y) \in \mathbb{R}^2 : \text{There exists some } n \in \mathbb{Z} \text{ such that } y = 2^n x\}$.
   (a) Verify that T is reflexive.
   (b) Verify that T is transitive.
   (c) Verify that T is an equivalence relation in $\mathbb{R}$.

2. Let p be a positive real number. Define the relation $R = (\mathbb{C}, \mathbb{C}, E)$ in $\mathbb{C}$ by

$$E = \left\{(\zeta, \eta) \in \mathbb{C}^2 : \text{ There exists some } n \in \mathbb{Z} \text{ such that } \eta = \zeta \cdot (\cos(np) + i\sin(np)). \right\}$$

   (a) Verify that R is reflexive.
   (b) Verify that R is transitive.
   (c) Is R an equivalence relation in $\mathbb{C}$? Justify your answer.

3. Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Define the relation $R = (\mathbb{C}^*, \mathbb{C}^*, G)$ in $\mathbb{C}^*$ by

$$G = \left\{(\zeta, \eta) \in (\mathbb{C}^*)^2 : \text{ There exists some } n \in \mathbb{Z} \text{ such that } \zeta = \eta \cdot 2^n(\cos(n) + i\sin(n)). \right\}.$$

   (a) Verify that R is reflexive.
   (b) Verify that R is transitive.
   (c) Is R an equivalence relation in $\mathbb{C}^*$? Justify your answer.

4. Define the relation $T = (\mathbb{R}, \mathbb{R}, G)$ in $\mathbb{R}$ by
   $G = \{(x, y) \mid x \in \mathbb{R} \text{ and } y \in \mathbb{R} \text{ and (there exists some } m, n \in \mathbb{Q} \text{ such that } y = 3^m 5^n x)\}$.
   (a) Verify that T is reflexive.
   (b) Verify that T is transitive.
   (c) Verify that T is an equivalence relation in $\mathbb{R}$.

5. Let A be a set, $G = \{(S, T) \mid S \in \mathfrak{P}(A) \text{ and } T \in \mathfrak{P}(A) \text{ and } S \subset T\}$ and $R = (\mathfrak{P}(A), \mathfrak{P}(A), G)$.
   (a) Verify that R is a partial ordering.
   (b) Suppose A has at least two distinct elements. Verify that R is not a total ordering.

6. (a) Let A be the set of all real-valued continuous functions on $[0, 1]$. Define the relation $S = (A, A, G)$ in A by

$$G = \left\{(f, g) \in A^2 : \int_0^x uf(u)du \leq \int_0^x ug(u)du \text{ for any } x \in [0, 1]\right\}.$$

   Is S a partial ordering in A? Justify your answer.

   (b) Let B be the set of all real-valued piecewise-continuous functions on $[0, 1]$. Define the relation $T = (B, B, H)$ in B by

$$H = \left\{(f, g) \in B^2 : \int_0^x uf(u)du \leq \int_0^x ug(u)du \text{ for any } x \in [0, 1]\right\}.$$

   Is T a partial ordering in B? Justify your answer.

7. Define the relation $S = (\mathbb{N}^2, \mathbb{N}^2, P)$ in $\mathbb{N}^2$ by $P = \left\{ (u, v) \;\middle|\; \begin{array}{l} \text{There exist } m, n, p, q \in \mathbb{N} \text{ such that} \\ u = (m, n), v = (p, q) \text{ and } \dfrac{2n+1}{2^m} \leq \dfrac{2q+1}{2^p} \end{array} \right\}$.

   Here $\leq$ is the usual ordering in $\mathbb{R}$.
   (a) Verify that S is a partial ordering in $\mathbb{N}^2$.
   (b) Is S a total ordering in $\mathbb{N}^2$? Why?

8. Define the relation $R = (\mathbb{C}, \mathbb{C}, P)$ by $P = \left\{ (\zeta, \eta) \;\middle|\; \begin{array}{l} \zeta, \eta \in \mathbb{C} \text{ and} \\ (\mathsf{Re}(\zeta) < \mathsf{Re}(\eta) \text{ or } (\mathsf{Re}(\zeta) = \mathsf{Re}(\eta) \text{ and } \mathsf{Im}(\zeta) \leq \mathsf{Im}(\eta))) \end{array} \right\}$.

(a) Let $\zeta, \eta \in \mathbb{C}$.

i. Verify that $(\zeta, \eta) \in P$ iff $(\mathsf{Re}(\zeta) \leq \mathsf{Re}(\eta)$ and $(\mathsf{Re}(\zeta) < \mathsf{Re}(\eta)$ or $\mathsf{Im}(\zeta) \leq \mathsf{Im}(\eta)))$.

ii. Verify that $(\zeta, \eta) \notin P$ iff $(\mathsf{Re}(\eta) < \mathsf{Re}(\zeta)$ or $(\mathsf{Re}(\eta) \leq \mathsf{Re}(\zeta)$ and $\mathsf{Im}(\eta) < \mathsf{Im}(\zeta)))$.

(b) Verify that R is a total ordering in $\mathbb{C}$.

Remark. Such a total ordering in $\mathbb{C}$ is known as a **lexicographical ordering**. Think of each complex number as a word with two 'letters', the first 'letter' being its real part and the second 'letter' being its imaginary part respectively. Now how do you arrange such 'two-letter words' in a dictionary?

9. Denote by Σ the set of all infinite sequences in $\mathbb{R}$. (Recall that each infinite sequence in $\mathbb{R}$ is a function from $\mathbb{N}$ to $\mathbb{R}$.) Let $k \in \mathbb{N}$. Define the relation $R_k = (\Sigma, \Sigma, E)$ by

$$E = \left\{ (\alpha, \beta) \;\middle|\; \begin{array}{l} \alpha, \beta \in \Sigma \text{ and there exist some } N \in \mathbb{N},\, C \geq 0 \\ \text{such that } (|\alpha(x) - \beta(x)| \leq C/x^k \text{ for any } x \geq N). \end{array} \right\}$$

(a) Verify that R_k is reflexive and symmetric.

(b) Verify that R_k is an equivalence relation in Σ.

10. (a) Let $A = \{0, 1, 2\}$, $G = \{(0, 0), (1, 1), (2, 2), (0, 1), (1, 2)\}$, and $R = (A, A, G)$. (Here $0, 1, 2$ are pairwise distinct objects.)

i. Verify that R is not symmetric.

ii. Verify that R is not transitive.

iii. Verify that R is reflexive.

(b) Let $B = \{0, 1\}$, $H = \{(0, 0), (0, 1), (1, 0)\}$, and $S = (B, B, H)$. (Here $0, 1$ are distinct objects.)

i. Verify that S is not reflexive.

ii. Verify that S is not transitive.

iii. Verify that S is symmetric.

(c) Let $C = \{0, 1, 2\}$, $J = \{(0, 1), (1, 2), (0, 2)\}$, and $T = (C, C, J)$. (Here $0, 1, 2$ are pairwise distinct objects.)

i. Verify that T is not reflexive.

ii. Verify that T is not symmetric.

iii. Verify that T is transitive.

Remark. Can you construct a relation in a non-empty set which is reflexive and symmetric but not transitive? Can you construct a relation in a non-empty set which is reflexive and transitive but not symmetric? Can you construct a relation in a non-empty set which is symmetric and transitive but not reflexive?

11. Dis-prove each of the statements below by giving an appropriate counter-example.

(a) *Let A be a non-empty set, and R be a relation in A. Suppose R is reflexive and symmetric. Then R is transitive.*

(b) *Let A be a non-empty set, and R be a relation in A. Suppose R is reflexive and transitive. Then R is symmetric.*

(c) *Let A be a non-empty set, and R be a relation in A. Suppose R is symmetric and transitive. Then R is reflexive.*

12. (a) Let A be a non-empty set, and R be a relation in A with graph G. Suppose R is symmetric and transitive.

Prove that the statements below are logically equivalent:

($\sharp$) *For any $x \in A$, there exists some $y \in A$ such that $(x, y) \in G$.*

($\flat$) *R is reflexive.*

(b) Let A be a non-empty set, and R be a relation in A with graph G. Suppose R is reflexive.

Prove that the statements below are logically equivalent:

($\sharp$) *For any $x, y, z \in A$, if $(x, y) \in G$ and $(y, z) \in G$ then $(z, x) \in G$.*

($\flat$) *R is symmetric and transitive.*

(c) Let A be a non-empty set, and R be a relation in A with graph G. Suppose R is reflexive.

Prove that the statements below are logically equivalent:

($\sharp$) *For any $x, y, z \in A$, if $(x, y) \in G$ and $(x, z) \in G$ then $(y, z) \in G$.*

($\flat$) *R is symmetric and transitive.*

13. Let A be a set, F be a subset of A^2, and $f = (A, A, F)$. Suppose f is a function from A to A. (Also think of f as a relation in A.) Prove the statements below:

   (a) *If f is reflexive as a relation in A then $f = \mathsf{id}_A$.*

   (b) *If f is transitive as a relation in A then $f \circ f = f$ as functions.*

   (c) *If f is transitive as a relation in A and f is injective as a function then $f = \mathsf{id}_A$.*

   (d) *If f is both symmetric and transitive as a relation in A then $f = \mathsf{id}_A$.*

14. We introduce the definition below:

   - *Let A, B be sets, $f : A \longrightarrow B$ be a function, and Q be a relation in B with graph H.*
     Define the subset f^*H of A^2 by $f^*H = \{\ (x, w) \mid x \in A \text{ and } w \in A \text{ and } (f(x), f(w)) \in H\ \}$.
     The relation (A, A, f^*H) is called **pull-back relation** *of Q by f. It is denoted by f^*Q in A.*

   Let A, B be sets, $f : A \longrightarrow B$ be a function, and Q be a relation in B with graph H.
   Prove the statements below:

   (a) *Suppose Q is reflexive. Then f^*Q is reflexive.*

   (b) *Suppose Q is symmetric. Then f^*Q is symmetric.*

   (c) *Suppose Q is transitive. Then f^*Q is transitive.*

   (d) *Suppose Q is an equivalence relation. Then f^*Q is an equivalence relation.*

   (e) *Suppose f^*Q is an equivalence relation and f is surjective. Then Q is an equivalence relation.*

   (f) *Suppose Q is reflexive and f^*Q is anti-symmetric. Then f is injective.*

   (g) *Suppose Q is a partial ordering and f is injective. Then f^*Q is a partial ordering.*

15. Let A be a non-empty set, and R be a relation in A with graph E.

   For any $x \in A$, we define $R[x] = \{y \in A : (x, y) \in E\}$. We define $\Omega = \{\ R[x]\ \big|\ x \in A\ \}$.

   Suppose that R is an equivalence relation in A.

   (a) Prove the statements below:

      i. *For any $x \in A$, $x \in R[x]$.*
      ii. $\emptyset \notin \Omega$.
      iii. *For any $x, y \in A$, if $(x, y) \in E$ then $R[y] \subset R[x]$.*
      iv. *For any $x, y \in A$, the statements ($\sharp$), ($\natural$), ($\flat$) are logically equivalent:*

      ($\sharp$) $(x, y) \in E$. ($\natural$) $R[x] = R[y]$. ($\flat$) $R[x] \cap R[y] \neq \emptyset$.

      Remark. $R[x]$ is called the **equivalence class** of x under the equivalence relation R.

   (b) Apply part (a), or otherwise, to prove that Ω is a partition of A, in the sense that the statements (N), (U), (D) are true:

      (N) $\emptyset \notin \Omega$.
      (U) $\{z \in A : z \in S \text{ for some } S \in \Omega\} = A$.
      (D) *For any $S, T \in \Omega$, exactly one of the statements '$S = T$', '$S \cap T = \emptyset$' is true.*

      Remark. We call Ω the **quotient** of A by the equivalence relation R, and usually write Ω as A/R. We refer to the elements of Ω as the equivalence classes under R.

   (c) Let Φ be the subset of $A \times \Omega$ given by $\Phi = \{\ (x, S)\ \big|\ x \in A \text{ and } S \in \Omega \text{ and } x \in S\ \}$. Define the relation $\varphi = (A, \Omega, \Phi)$.

      i. Prove that φ is a surjective function, and that $\varphi(x) = R[x]$ for any $x \in A$.
      Remark. We call φ the **quotient mapping** of the equivalence relation R.
      ii. Let B be a set and $f : A \longrightarrow B$ be a function. Suppose that for any $x, y \in A$, if $(x, y) \in E$ then $f(x) = f(y)$. Prove that there exists some unique function $g : \Omega \longrightarrow B$ such that $g \circ \varphi = f$.

16. Define the relation $R = (\mathbb{C}, \mathbb{C}, E)$ in $\mathbb{C}$ by $E = \{(\zeta, \eta) \in \mathbb{C}^2 : \mathsf{Re}(\zeta) = \mathsf{Re}(\eta)\}$.

   (a) Verify that R is reflexive.

(b) Verify that R is symmetric.

(c) Verify that R is an equivalence relation in $\mathbb{C}$.

(d) For any $\zeta \in \mathbb{C}$, denote by $[\zeta]$ the equivalence class of ζ under R.
(Note that by definition, $[\zeta] = \{\eta \in \mathbb{C} : (\zeta, \eta) \in E\}$.)
What are the respective equivalence classes of 1, 0, i under R? Describe these sets in geometric terms in the Argand plane.

17. Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$, $\mathbb{R}^* = \mathbb{R}\backslash\{0\}$.

Define the relation $R = (\mathbb{C}^*, \mathbb{C}^*, E)$ in $\mathbb{C}^*$ by $E = \left\{(\zeta, \eta) \in (\mathbb{C}^*)^2 : \dfrac{\mathsf{Re}(\zeta)}{|\zeta|^2} = \dfrac{\mathsf{Re}(\eta)}{|\eta|^2}\right\}$.

(a) Verify that R is an equivalence relation in $\mathbb{C}^*$.

(b) For any $\zeta \in \mathbb{C}^*$, denote by $[\zeta]$ the equivalence class of ζ under R.

i. Let $a \in \mathbb{R}^*$. Verify that $[ai] = \{ti \mid t \in \mathbb{R}^*\}$.

ii. Let $\zeta \in \mathbb{C}^*$. Suppose $\mathsf{Re}(\zeta) \neq 0$. Define $r_\zeta = \dfrac{|\zeta|^2}{2\mathsf{Re}(\zeta)}$. Verify the statements (†) and (‡):

(†) $(\zeta, 2r_\zeta) \in E$.

(‡) *Suppose $\eta \in \mathbb{C}^*$. Then $\eta \in [\zeta]$ iff $(\mathsf{Re}(\eta) - r_\zeta)^2 + (\mathsf{Im}(\eta))^2 = (r_\zeta)^2$.*

18. Define the relation $T = (\mathbb{C}, \mathbb{C}, G)$ in $\mathbb{C}$ by $G = \{(\zeta, \eta) \in \mathbb{C}^2 : \zeta^4 = \eta^4\}$.

(a) Verify that T is an equivalence relation in $\mathbb{C}$.

(b) For any $\zeta \in \mathbb{C}$, denote by $[\zeta]$ the equivalence class of ζ under T.
Prove the statements below:

i. *For any $\zeta, \eta \in \mathbb{C}$, if $\eta \in [\zeta]$ then ($\eta = \zeta$ or $\eta = i\zeta$ or $\eta = -\zeta$ or $\eta = -i\zeta$).*

ii. *For any $\zeta \in \mathbb{C}$, $[\zeta] = \{\zeta, i\zeta, -\zeta, -i\zeta\}$.*

(c) Denote by Ω the quotient of $\mathbb{C}$ by T, and define the function $\pi : \mathbb{C} \longrightarrow \Omega$ by $\pi(\zeta) = [\zeta]$ for any $\zeta \in \mathbb{C}$.
Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function. Define

$$\varphi = \left\{(U, \chi) \,\middle|\, \begin{array}{l} U \in \Omega \text{ and } \chi \in \mathbb{C} \text{ and} \\ \text{there exists } \zeta \in \mathbb{C} \text{ such that } U = [\zeta] \text{ and } \chi = f(\zeta^4). \end{array}\right\}.$$

Note that $\varphi \subset \Omega \times \mathbb{C}$.
Prove the statements below:

i. *φ is a function from Ω to $\mathbb{C}$.*

ii. *$(\varphi \circ \pi)(\zeta) = f(\zeta^4)$ for any $\zeta \in \mathbb{C}$.*

iii. *Let $\psi : \Omega \longrightarrow \mathbb{C}$ is a function. Suppose $(\psi \circ \pi)(\zeta) = f(\zeta^4)$ for any $\zeta \in \mathbb{C}$. Then $\psi = \varphi$.*

19. Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a surjective function.
Define the relation $R_f = (A, A, E_f)$ in A by $E_f = \{(x, y) \mid x, y \in A \text{ and } f(x) = f(y)\}$.

(a) Verify that R_f is an equivalence relation.

(b) For any $x \in A$, denote the equivalence class of x under R_f by $[x]_f$.
Verify that $[x]_f = f^{-1}(\{f(x)\})$ for any $x \in A$.

(c) Define $\Omega = \{S \in \mathfrak{P}(A) \mid S = [x]_f \text{ for some } x \in A\}$.
Verify that Ω is a partition of A, in the sense that the statements (N), (U), (D) are true:

(N) $\emptyset \notin \Omega$.

(U) $\{z \in A : z \in S$ *for some* $S \in \Omega\} = A$.

(D) *For any $S, T \in \Omega$, exactly one of the statements '$S = T$', '$S \cap T = \emptyset$' is true.*

(d) Define $G_f = \{(x, S) \mid x \in A \text{ and } S \in \Omega \text{ and } x \in S\}$ and $\pi_f = (A, \Omega, G_f)$.
Verify that π_f is a surjective function.

(e) Let $\varphi : A \longrightarrow C$ be a function. Suppose that for any $x, y \in A$, if $f(x) = f(y)$ then $\varphi(x) = \varphi(y)$. Prove that there exists some unique function $\psi : \Omega \longrightarrow C$ such that $\psi \circ \pi = \varphi$.

20. Recall that whenever $n \in \mathbb{N}\backslash\{0,1\}$, the relation $R_n = (\mathbb{Z}, \mathbb{Z}, E_n)$ given by $E_n = \{(x,y) \mid x, y \in \mathbb{Z} \text{ and } x \equiv y (\text{mod } n)\}$ is an equivalence relation in $\mathbb{Z}$. The quotient of $\mathbb{Z}$ by R_n is the set $\mathbb{Z}_n$.

For each $x \in \mathbb{Z}$, we denote by $[x]_n$ the equivalence class of x under the equivalence relation R_n in $\mathbb{Z}$. It is the element of $\mathbb{Z}_n$ given explicitly by $[x]_n = \{x \in \mathbb{Z} : (x,y) \in E_n\} = \{x \in \mathbb{Z} : x \equiv y(\text{mod } n)\}$.

Below are several 'declarations' through each of which some function is supposed to be defined. Determine whether it makes sense or not. Justify your answer.

(a) '*Define the function* $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}$ *by* $f([k]_{10}) = 10k$ *for any* $k \in \mathbb{Z}$.'

(b) '*Define the function* $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{100}$ *by* $f([k]_{10}) = [k]_{100}$ *for any* $k \in \mathbb{Z}$.'

(c) '*Define the function* $f : \mathbb{Z}_{100} \longrightarrow \mathbb{Z}_{10}$ *by* $f([k]_{100}) = [k]_{10}$ *for any* $k \in \mathbb{Z}$.'

(d) '*Define the function* $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{100}$ *by* $f([k]_{10}) = [10k]_{100}$ *for any* $k \in \mathbb{Z}$.'

(e) '*Define the function* $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{10}$ *by* $f([k]_{10}) = [3k]_{10}$ *for any* $k \in \mathbb{Z}$.'

(f) '*Define the function* $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{10}$ *by* $f([3k]_{10}) = [k]_{10}$ *for any* $k \in \mathbb{Z}$.'

(g) '*Define the function* $f : \mathbb{Z}_{10} \longrightarrow \mathbb{Z}_{10}$ *by* $f([4k]_{10}) = [3k]_{10}$ *for any* $k \in \mathbb{Z}$.'

21. Let $\mathbb{G} = \{\zeta \in \mathbb{C} : \text{Re}(\zeta) \in \mathbb{Z} \text{ and } \text{Im}(\zeta) \in \mathbb{Z}\}$. ($\mathbb{G}$ is the set of all Gaussian integers.)

Define the subset E of $\mathbb{C}^2$ by $E = \{(\zeta, \eta) \mid \zeta, \eta \in \mathbb{C} \text{ and } \zeta - \eta \in \mathbb{G}\}$.

Define $R = (\mathbb{C}, \mathbb{C}, E)$.

For each $\zeta \in \mathbb{C}$, define $[\zeta] = \{\eta \in \mathbb{C} : (\zeta, \eta) \in E\}$.

Let $T = \{[\zeta] \mid \zeta \in \mathbb{C}\}$.

Throughout this question, you may take the validity of the statements **(S1)**, **(S2)**, **(S3)** *for granted:*

(S1) *R is an equivalence relation in $\mathbb{C}$.*

(S2) *For any $\zeta \in \mathbb{C}$, $\zeta \in [\zeta]$.*

(S3) *For any $\zeta, \eta \in \mathbb{C}$, the statements ($\sharp$), ($\natural$), ($\flat$) are equivalent:*

($\sharp$) $(\zeta, \eta) \in E$. ($\natural$) $[\zeta] = [\eta]$. ($\flat$) $[\zeta] \cap [\eta] \neq \emptyset$.

(a) Define the subset Σ of $T^2 \times T$ by

$$\Sigma = \left\{ ((p,q),r) \;\middle|\; \begin{array}{l} p,q,r \in T \text{ and (there exist some } \zeta, \eta \in \mathbb{C} \\ \text{such that } p = [\zeta], q = [\eta] \text{ and } r = [\zeta + \eta]). \end{array} \right\}.$$

Define $\alpha = (T^2, T, \Sigma)$. Note that α is a relation from T^2 to T.

Verify that α is a function from T^2 to T.

(b) Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a surjective function. Consider the statements $(\star), (\star\star)$ below:

$(\star)$ *There exists some surjective function $h : T \longrightarrow T$ such that for any $\zeta \in \mathbb{C}$, $h([\zeta]) = [f(\zeta)]$.*

$(\star\star)$ *For any $\zeta, \eta \in \mathbb{C}$, if $\zeta - \eta \in \mathbb{G}$ then $f(\zeta) - f(\eta) \in \mathbb{G}$.*

i. Suppose $(\star)$ holds. Prove that $(\star\star)$ holds.

ii. Suppose $(\star\star)$ holds. Prove that $(\star)$ holds.

22. Let $\lambda \in \mathbb{C}\backslash\{0\}$.

Define the subset E of $\mathbb{C}^2$ by $E = \{(\zeta, \eta) \in \mathbb{C}^2 : \text{Re}(\overline{\lambda}\zeta) = \text{Re}(\overline{\lambda}\eta)\}$.

Define $R = (\mathbb{C}, \mathbb{C}, E)$.

For each $\zeta \in \mathbb{C}$, define $[\zeta] = \{\eta \in \mathbb{C} : (\zeta, \eta) \in E\}$.

Let $L = \{[\zeta] \mid \zeta \in \mathbb{C}\}$.

Throughout this question, you may take the validity of the statements **(S1)**, **(S2)**, **(S3)** *for granted:*

(S1) *R is an equivalence relation in $\mathbb{C}$.*

(S2) *For any $\zeta \in \mathbb{C}$, $\zeta \in [\zeta]$.*

(S3) *For any $\zeta, \eta \in \mathbb{C}$, the statements ($\sharp$), ($\natural$), ($\flat$) are equivalent:*

($\sharp$) $(\zeta, \eta) \in E$. ($\natural$) $[\zeta] = [\eta]$. ($\flat$) $[\zeta] \cap [\eta] \neq \emptyset$.

(a) Define the subset Σ of $L^2 \times L$ by

$$\Sigma = \left\{ ((p,q),r) \;\middle|\; \begin{array}{l} p,q,r \in L \text{ and (there exist some } \zeta,\eta \in \mathbb{C} \\ \text{such that } p = [\zeta], q = [\eta] \text{ and } r = [\zeta+\eta]). \end{array} \right\}.$$

Define $\alpha = (L^2, L, \Sigma)$. Note that α is a relation from L^2 to L.

Verify that α is a function from L^2 to L.

(b) Now also suppose $\mathsf{Re}(\lambda) \neq 0$. Define the function $f : \mathbb{C} \longrightarrow \mathbb{R}$ by

$$f(\zeta) = \frac{\mathsf{Re}(\overline{\lambda}\zeta)}{\mathsf{Re}(\lambda)} \quad \text{for any } \zeta \in \mathbb{C}.$$

Prove the statement $(\star)$:

$(\star)$ *There exists some bijective function $h : L \longrightarrow \mathbb{R}$ such that (for any $\zeta \in \mathbb{C}$, $h([\zeta]) = f(\zeta)$) and (for any $\sigma, \tau \in \mathbb{C}$, $h(\alpha([\sigma],[\tau])) = f(\sigma) + f(\tau)$).*

23. Write $\mathbb{Z}^* = \mathbb{Z} \backslash \{0\}$.

Define the subset F of $(\mathbb{Z} \times \mathbb{Z}^*)^2$ by

$$F = \{((x,y),(x',y')) \mid x, x' \in \mathbb{Z} \text{ and } y, y' \in \mathbb{Z}^* \text{ and } xy' = x'y \}.$$

Define $Q = (\mathbb{Z} \times \mathbb{Z}^*, \mathbb{Z} \times \mathbb{Z}^*, F)$

For any $x \in \mathbb{Z}, y \in \mathbb{Z}^*$, define $[x,y] = \{(s,t) \mid s \in \mathbb{Z} \text{ and } t \in \mathbb{Z}^* \text{ and } ((x,y),(s,t)) \in F\}$.

Let $\Phi = \{[x,y] \mid x \in \mathbb{Z} \text{ and } y \in \mathbb{Z}^*\}$.

Throughout this question, you may take the validity of the statements **(S1)**, **(S2)**, **(S3)** *for granted:*

(S1) *Q is an equivalence relation in $\mathbb{Z} \times \mathbb{Z}^*$.*

(S2) *For any $x \in \mathbb{Z}$, for any $y \in \mathbb{Z}^*$, $(x,y) \in [(x,y)]$.*

(S3) *For any $x, x' \in \mathbb{Z}$, for any $y, y' \in \mathbb{Z}^*$, the statements $(\sharp)$, $(\natural)$, $(\flat)$ are equivalent:*

$(\sharp)$ $((x,y),(x',y')) \in F$. $\quad (\natural)$ $[x,y] = [x',y']$. $\quad (\flat)$ $[x,y] \cap [x',y'] \neq \emptyset$.

(a) Define the subset G of $\Phi^2 \times \Phi$ by

$$G = \left\{ ((u,v),w) \;\middle|\; \begin{array}{l} \text{There exist some } x, x' \in \mathbb{Z}, y, y' \in \mathbb{Z}^* \\ \text{such that } u = [x,y] \text{ and } v = [x',y'] \text{ and } w = [xy' + yx', yy']. \end{array} \right\}.$$

Define $\alpha = (\Phi^2, \Phi, G)$. Note that α is a relation from G^2 to G.

Verify that α is a function.

(b) For any $u, v \in \Phi$, we write $\alpha(u,v)$ as $u \oplus v$.

Verify the statements below:

i. *For any $u, v \in \Phi$, $u \oplus v = v \oplus u$.*

ii. *For any $u, v, w \in \Phi$, $(u \oplus v) \oplus w = u \oplus (v \oplus w)$.*

iii. *There exists some unique $e \in \Phi$ such that for any $u \in \Phi$, $u \oplus e = u$ and $e \oplus u = u$.*

iv. *For any $u \in \Phi$, there exists some unique $v \in \Phi$ such that $u \oplus v = e$ and $v \oplus u = e$.* (Here e is the unique element of Φ which satisfies $u \oplus e = u = e \oplus u$ for any $u \in \Phi$.)