MATH1050 Partial orderings defined by the subset relation, and Zorn's Lemma.

0. (a) This handout is a continuation of the Handout *Partial orderings, total orderings* and the Handout *Well-order relations and the Well-ordering Principle*.

   (b) Although the theoretical tools (definitions and results) introduced in this Handout are formulated for the general situation, our focus will be on their application in partial orderings defined by subset relation.

1. **Definition.**

   Let A be a set, and T be a partial ordering in A with graph G. Write $u \preceq v$ exactly when $(u, v) \in G$.

   Let B be a subset of A.

   Let $\lambda \in B$. We say λ is a $\left\{ \begin{array}{c} \textbf{greatest} \\ \textbf{least} \end{array} \right\}$ **element** *of B with respect to T if the statement (GL) hold:*

   $$(GL): \quad \text{For any } x \in B, \left\{ \begin{array}{c} x \preceq \lambda \\ x \succeq \lambda \end{array} \right\}.$$

   Remark. A subset of A has at most one greatest/least element with respect to T. Hence it makes sense to refer to such an element of A as 'the' greatest/least element with respect to T, if it exists.

   Further remark. As stated in the Handout *Partial orderings, total orderings and well-order relations*, the notion of greatest/least element for arbitrary sets with respect to general partial orderings is generalized from the notion of greatest/least element for subsets of $\mathbb{R}$.

2. **Terminologies on partial orderings defined by the subset relation.**

   Suppose M is a set.

   Recall that the partial ordering in $\mathfrak{P}(M)$ defined by the subset relation is the relation in $\mathfrak{P}(M)$ with graph $\{(U, V) \mid U, V \in \mathfrak{P}(M) \text{ and } U \subset V\}$.

   Suppose A is a subset of $\mathfrak{P}(M)$. The restriction of the above partial ordering to A will be referred to as the partial ordering '**subset relation inside** A'.

3. **Example (a1).**

   Regard $0, 1, 2, 3$ as distinct objects.

   Let $m = \{2\}$, $n = \{1, 2\}$, $o = \{2, 3\}$, $p = \{0, 2, 3\}$, $q = \{1, 2, 3\}$, $r = \{0, 1, 2, 3\}$.

   (a) Let $A_1 = \{m, n, q, r\}$.
   A_1 is totally ordered by the subset relation inside A_1.
   m is the least element with respect to the subset relation inside A_1.
   r is the greatest element with respect to the subset relation inside A_1.

   (b) Let $A_2 = \{m, n, o, p, q, r\}$.
   The subset relation inside A can be visualized through a display of 'all' possible 'chains of inequalities' for the elements of A below:

   - $m \subset n \subset q \subset r$,
   - $m \subset o \subset p \subset r$,
   - $m \subset o \subset q \subset r$.

   A_2 is not totally ordered by the subset relation inside A_2.
   m is the least element with respect to the subset relation inside A_2.
   r is the greatest element with respect to the subset relation inside A_2.

4. **Example (b).**

   Regard $0, 1, 2, 3$ as distinct objects.

   Let $m = \{0\}$, $n = \{1\}$, $o = \{3\}$, $p = \{0, 1\}$, $q = \{1, 2\}$, $r = \{0, 1, 2\}$, $s = \{0, 1, 3\}$, $t = \{1, 2, 3\}$.

   Let $A = \{m, n, o, p, q, r, s, t\}$.

   The subset relation inside A can be visualized through a display of 'all' possible 'chains of inequalities' for the elements of A below:

- $m \subset p \subset r$,
- $m \subset p \subset s$,
- $n \subset p \subset r$,
- $n \subset p \subset s$,
- $n \subset q \subset r$,
- $n \subset q \subset t$,
- $o \subset s$,
- $o \subset t$.

(a) A has no least element with respect to the subset relation inside A. Reason:

- If it were true that λ was a least element, then $\lambda \subset x$ for any $x \in A$.
  However, for any $u \in A$, there exists some $v \in A$ such that $u \not\subset v$.

Similarly A has no greatest element with respect to the subset relation inside A.

(b) For each $x \in A$, we define the set $\mathrm{C}[x]$ by $\mathrm{C}[x] = \{y \in A : x \subset y \text{ or } y \subset x\}$,

We may call $\mathrm{C}[x]$ the set of comparable's with x in A. ($y \in \mathrm{C}[x]$ iff x, y are comparable with respect to the subset relation inside A.)

i. Note that $\mathrm{C}[m] = \{m, p, r, s\}$, and $m \subset p \subset r$, $m \subset p \subset s$.
Then m is the least element of $\mathrm{C}[m]$ with respect to the subset relation inside A.
What happens can be described in plain words as:
   - Amongst elements of A which are comparable to m with respect to the subset relation inside A, none will precede m.

ii. Note that $\mathrm{C}[r] = \{m, n, p, q, r\}$, and $m \subset p \subset r$, $n \subset p \subset r$, $n \subset q \subset r$.
Then r is the greatest element of $\mathrm{C}[r]$ with respect to the subset relation inside A.
What happens can be described in plain words as:
   - Amongst elements of A which are comparable to r with respect to the subset relation inside A, none will succeed r.

The way m, r stand out in Example (b) illustrates the idea of the notions of maximality and minimality for partial orderings.

5. **Definition.**

Let A be a set, and T be a partial ordering in A with graph G. Write $u \preceq v$ exactly when $(u, v) \in G$.

Let B be a subset of A.

Let $\mu \in B$. We say μ is a $\left\{ \begin{array}{c} \textbf{maximal} \\ \textbf{minimal} \end{array} \right\}$ **element** *of B with respect to T if the statement (MM) holds:*

$$(MM): \quad \text{For any } x \in B, \text{ if } \left\{ \begin{array}{c} \mu \preceq x \\ \mu \succeq x \end{array} \right\} \text{ then } \mu = x.$$

Remark.

(a) The notion of greatest/least element and the notion of maximal/minimal element coincide for totally ordered sets (which we call chains), in which the Law of Trichotomy holds.

The notion of maximal/minimal element becomes relevant for partially ordered sets which are not totally ordered.

(b) The greatest/least element of a set B with respect to a partial ordering T is automatically the (unique) maximal/minimal element of B with respect to T.

(c) A maximal/minimal element of a set B with respect to a partial ordering T is not necessarily the greatest/least element of B with respect to T.

In fact, even when B has a maximal/minimal element with respect to T, B does not necessarily have a greatest/least element with respect to T.

6. **Example (b) re-visited.**

Regard $0, 1, 2, 3$ as distinct objects.

Let $m = \{0\}$, $n = \{1\}$, $o = \{3\}$, $p = \{0, 1\}$, $q = \{1, 2\}$, $r = \{0, 1, 2\}$, $s = \{0, 1, 3\}$, $t = \{1, 2, 3\}$.

Let $A = \{m, n, o, p, q, r, s, t\}$.

The subset relation inside A can be visualized through a display of 'all' possible 'chains of inequalities' for the elements of A below:

- $m \subset p \subset r$,
- $m \subset p \subset s$,
- $n \subset p \subset r$,
- $n \subset p \subset s$,
- $n \subset q \subset r$,
- $n \subset q \subset t$,
- $o \subset s$,
- $o \subset t$.

A is not totally ordered by the subset relation inside A.

A has no least element with respect to the subset relation inside A.

A has no greatest element with respect to the subset relation inside A.

m, n, o are minimal elements of A with respect to the subset relation inside A.

r, s, t are maximal elements of A with respect to the subset relation inside A.

7. **Example (c).**

Regard $0, 1$ as distinct objects. Let $M = \{0, 1\}$.

(a) Note that $\mathfrak{P}(M) = \{\emptyset, \{0\}, \{1\}, M\}$.

The subset relation inside $\mathfrak{P}(M)$ is not a total ordering. (Reason: $\{0\} \not\subset \{1\}$ and $\{1\} \not\subset \{0\}$.)

$\mathfrak{P}(M)$ has a greatest element with respect to the subset relation iside $\mathfrak{P}(M)$. It is M.

$\mathfrak{P}(M)$ has a least element with respect to the subset relation iside $\mathfrak{P}(M)$. It is $\emptyset$.

(b) Define the subsets S_1, S_2, S_3, S_4 of $\mathfrak{P}(M)$ by

$$S_1 = \{\emptyset, \{0\}, M\}, \quad S_2 = \{\emptyset, \{0\}, \{1\}\}, \quad S_3 = \{\{0\}, \{1\}, M\}, \quad S_4 = \{\{0\}, \{1\}\}.$$

We list the respective least element, greatest element, maximal elements, minimal elements (if exist) of each of S_1, S_2, S_3, S_4 with respect to the subset relation inside $\mathfrak{P}(M)$:

	least element?	minimal elements?	greatest element?	maximal elements?
S_1	$\emptyset$	$\emptyset$	M	M
S_2	$\emptyset$	$\emptyset$	*nil*	$\{0\}$, $\{1\}$
S_3	*nil*	$\{0\}$, $\{1\}$	M	M
S_4	*nil*	$\{0\}$, $\{1\}$	*nil*	$\{0\}$, $\{1\}$

8. **Example (d).**

For each $x \in \mathbb{Z}$, define $\langle x \rangle = \{y \in \mathbb{Z} : y \text{ is divisible by } x\}$. The set $\langle x \rangle$ is called the ideal in $\mathbb{Z}$ generated by the integer x.

Note that $\langle 0 \rangle = \{0\}$ and $\langle 1 \rangle = \langle -1 \rangle = \mathbb{Z}$. For each $x \in \mathbb{Z} \backslash \{-1, 0, 1\}$, $\langle x \rangle$ is the set of all integral multiples of x.

Let $A = \{S \in \mathfrak{P}(\mathbb{Z}) : S = \langle x \rangle \text{ for some } x \in \mathbb{Z}\}$.

The subset relation inside A is not a total ordering.

A has a greatest element with respect to the subset relation inside A. This is $\langle 1 \rangle$. It is also the unique maximal element of A with respect to the subset relation inside A.

A has no least element with respect to the subset relation inside A.

There is one minimal element of A with respect to the subset relation inside A. This is $\langle 0 \rangle$.

Remark. What seems to prevent A from having any minimal element other than $\langle 0 \rangle$ is the lack of 'boundedness'. Such a lack of 'boundedness' allows for a non-terminating chain of strict inequalities, obtained by 'shrinking' an arbitrary element of $A \backslash \{\langle 0 \rangle\}$, say, $\langle x \rangle$, in which x is a non-zero integer, to 'smaller and smaller' elements of A:

$$\langle x \rangle \supsetneq \langle 2x \rangle \supsetneq \langle 4x \rangle \supsetneq \langle 8x \rangle \supsetneq \cdots \supsetneq \langle 2^n x \rangle \supsetneq \langle 2^{n+1} x \rangle \supsetneq \cdots$$

9. **Example (e).**

Let $A = \left\{ I \in \mathfrak{P}(\mathbb{R}) : \begin{array}{l} \text{There exist some } s, t \in \mathbb{R} \text{ such that} \\ s + 1 \leq t \leq s + 2, \text{ and } I = [s, t]. \end{array} \right\}$.

(In plain words, A is the set of all closed intervals in $\mathbb{R}$ of length at least 1 and at most 2.)

The subset relation inside A is not a total ordering.

A has no least element with respect to the subset relation inside A.

A has no greatest element with respect to the subset relation inside A.

The minimal elements of A with respect to the subset relation inside A are exactly all closed interval of length 1.

The maximal elements of A with respect to the subset relation inside A are exactly all closed interval of length 2.

10. **Example (f).**

Let $A = \left\{ I \in \mathfrak{P}(\mathbb{R}) : \begin{array}{l} \text{There exist some } s, t \in \mathbb{R} \text{ such that} \\ s+1 < t < s+2, \text{ and } I = [s,t]. \end{array} \right\}$.

(In plain words, A is the set of all closed intervals in $\mathbb{R}$ of length greater than 1 and less than 2.)

The subset relation inside A is not a total ordering.

A has no least element with respect to the subset relation inside A.

A has no greatest element with respect to the subset relation inside A.

A has no minimal elements with respect to the subset relation inside A. (Why? Fill in the detail.)

A has no maximal elements with respect to the subset relation inside A. Reason:

- Suppose I were a maximal element of A with respect to the subset relation inside A.
  By definition, there exists some $s, t \in \mathbb{R}$ such that $s+1 < t < s+2$ and $I = [s,t]$.
  Define $u = \dfrac{t+(s+2)}{2}$. Since $t < s+2$, we have $t < u < s+2$. Then $s+1 < t < u < s+2$.
  Define $J = [s,u]$. We have $J \in A$ and $I \subsetneqq J$. Contradiction arises.

Remark. What seems to prevent A from having maximal elements is the lack of 'boundedness'. Such a lack of 'boundedness' allows for a non-terminating chain of strict inequalities, obtained by 'expanding' an arbitrary element of $A\backslash\{\langle 0\rangle\}$, say, $[s, s+2-\delta]$, in which s is a real number and δ is a positive real number less than 2, to 'larger and larger' elements of A:

$$[s, s+2-\delta] \subsetneqq \left[s, s+2-\frac{\delta}{2}\right] \subsetneqq \left[s, s+2-\frac{\delta}{4}\right] \subsetneqq \left[s, s+2-\frac{\delta}{8}\right] \subsetneqq \cdots \subsetneqq \left[s, s+2-\frac{\delta}{2^n}\right] \subsetneqq \left[s, s+2-\frac{\delta}{2^{n+1}}\right] \subsetneqq \cdots$$

11. **Definition.**

Let A be a set, and T be a partial ordering in A with graph G. Write $u \preceq v$ exactly when $(u,v) \in G$.

Let B be a subset of A.

Let $\beta \in A$. We say β is a(n) $\left\{ \begin{array}{c} \textbf{upper} \\ \textbf{lower} \end{array} \right\}$ **bound of** B **in** A *with respect to T if the statement (UL) holds:*

(UL): *For any $x \in B$,* $\left\{ \begin{array}{c} x \preceq \beta \\ x \succeq \beta \end{array} \right\}$.

In this situation, we say that B is bounded $\left\{ \begin{array}{c} \textit{above} \\ \textit{below} \end{array} \right\}$ *in A by β with respect to T.*

Remark.

(a) The greatest/least element of a subset B of A with respect to T (if it exists) is automatically an upper/lower bound of B in A with respect to T.

(b) A subset of A which is bounded above/below in A with respect to T does not necessarily have a greatest/least element with respect to T.

Further remark. The notion of upper/lower bound for arbitrary sets with respect to general partial orderings is generalized from the notion of upper/lower bound for subsets of $\mathbb{R}$.

12. **Example (g).**

Let $A = \left\{ I \in \mathfrak{P}(\mathbb{R}) : \begin{array}{l} \text{There exist some } s, t \in \mathbb{R} \text{ such that} \\ s < t \leq s+1, \text{ and } I = [s,t]. \end{array} \right\}$.

(In plain words, A is the set of all closed intervals in $\mathbb{R}$ of length greater than 0 and of length at most 1.)

(a) A has (many) maximal elements with respect to the subset relation inside A. They are exactly the closed intervals of length 1.

(b) Coincidentally, it happens that every chain in A is bounded above in A with respect to the subset relation inside A. Reason:

- Suppose C is a chain with respect to the subset relation inside A.
  Define $K = \{s \in \mathbb{R} : \text{There exists some } I \in C,\ t \in \mathbb{R} \text{ such that } I = [s,t]\}$,
  $L = \{t \in \mathbb{R} : \text{There exists some } I \in C,\ s \in \mathbb{R} \text{ such that } I = [s,t]\}$, and $J = \bigcup_{I \in C} I$.
  After some work, we deduce that K has an infimum in $\mathbb{R}$, L has a supremum in $\mathbb{R}$, and J is an interval of length at most 1 whose left endpoint is the infimum of K in $\mathbb{R}$ and whose right endpoint is the supremum of L in $\mathbb{R}$.
  Denote by $\overline{J}$ the closed interval with the same endpoints of J. (This is the closure of J in $\mathbb{R}$.)
  Then $\overline{J} \in A$, and $I \subset J \subset \overline{J}$ for any $I \in C$.
  Therefore $\overline{J}$ is an upper bound of C in A with respect to the subset relation inside A.
  (Note that $\overline{J}$ is not necessarily an element of C. This is the point of its being an upper bound of C with respect to the subset relation inside A.)

Remark. This coincidence turns out to be not something by chance, if we choose to believe the validity of Zorn's Lemma.

13. **Zorn's Lemma.**

Let A be a set, and T be a partial ordering in A. Suppose every chain in A with respect to T is bounded above with respect to T. Then A has a maximal element with respect to T.

Theorem (1).

The statement of the Well-ordering Principle and the statement of Zorn's Lemma are logically equivalent.

Remark. Hence we can only choose between *believing* Zorn's Lemma and *not believing* it.

14. **Appendix: An application of Zorn's Lemma in linear algebra.**

Refer to the Handout *Spanning sets, linearly independent sets, and bases* for the definitions and basic results concerned with linear algebra.

We are going to apply Zorn's Lemma to prove that every vector space (over whatever field) has a base.

Theorem (2).

Suppose V be a vector space over a field $\mathbb{F}$. Then V has a base over $\mathbb{F}$.

Proof of Theorem (2), with the help of Zorn's Lemma.

Let V be a vector space over a field $\mathbb{F}$, and $\mathfrak{I}$ be the set of all linearly independent subsets of V over $\mathbb{F}$.

By definition, $\mathfrak{I}$ is a subset of $\mathfrak{P}(V)$, which is partially ordered by the subset relation inside $\mathfrak{I}$.

Here we verify that every chain in $\mathfrak{I}$ with respect to the subset relation inside $\mathfrak{I}$ is bounded above in $\mathfrak{I}$ with respect to the subset relation inside $\mathfrak{I}$.

Let C be a chain in $\mathfrak{I}$ with respect to the subset relation inside $\mathfrak{I}$.

Define $H = \bigcup_{S \in \mathfrak{C}} S$. By definition, H is a subset of V. We check that H is an upper bound of C in $\mathfrak{I}$ with respect to the subset relation inside $\mathfrak{I}$:

(a) For any $T \in C$, we have $T \subset H$. (So, if it indeed happens that $H \in \mathfrak{I}$, then B will be an upper bound of C in $\mathfrak{I}$with respect to the subset relation inside $\mathfrak{I}$.)

(b) [We now check that H is a linearly independent subset of V over $\mathbb{F}$.]
Pick any $k \in \mathbb{N}\backslash\{0\}$. Pick any $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in H$. Suppose $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k$ are pairwise distinct. Pick any $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{F}$.
Suppose $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = 0$.
For each $j \in [\![1, k]\!]$, there exists some $S_j \in C$ such that $\mathbf{v}_j \in S_j$.

Since C is a chain, there exists some $j_0 \in [\![1, k]\!]$ such that $S_j \subset S_{j_0}$ for any $j \in [\![1, k]\!]$.

Therefore, we have $\mathbf{v}_1, \mathbf{v}_2, \cdots, \mathbf{v}_k \in S_{j_0}$.

By definition, S_{j_0} is a linearly independent set over IF. Since $\alpha_1\mathbf{v}_1 + \alpha_2\mathbf{v}_2 + \cdots + \alpha_k\mathbf{v}_k = 0$, we have $\alpha_1 = \alpha_2 = \cdots = \alpha_k = 0$.

It follows that H is a linearly independent subset of V over IF. Hence H is an upper bound of C in $\mathfrak{J}$ with respect to the subset relation inside $\mathfrak{J}$.

We have verified that every chain in $\mathfrak{J}$ with respect to $\subset$ is bounded above in $\mathfrak{J}$ with respect to the subset relation inside $\mathfrak{J}$.

According to Zorn's Lemma, $\mathfrak{J}$ has a maximal element, say, M, with respect to the subset relation inside $\mathfrak{J}$.

By definition, M is a linearly independent subset of V over IF. We verify that $\mathsf{Span}_{\mathsf{IF}}(M) = V$:

By definition, we have $\mathsf{Span}_{\mathsf{IF}}(M) \subset V$.

Suppose it were false that $\mathsf{Span}_{\mathsf{IF}}(M) = V$. Then $\mathsf{Span}_{\mathsf{IF}}(M) \subsetneq V$. There would exist some $y \in V \backslash \{0\}$ such that $y \notin \mathsf{Span}_{\mathsf{IF}}(M)$.

By definition $y \notin M$.

Define $M' = M \cup \{y\}$. By definition, $M \subset M'$. Moreover $M' \in \mathfrak{J}$. (Why?)

Since M is a maximal element of $\mathfrak{J}$ with respect to the subset relation inside $\mathfrak{J}$, we have $M = M'$. But then $y \in M' = M$, which is a contradiction against the statement '$y \notin M$'.

Hence we have $\mathsf{Span}_{\mathsf{IF}}(M) = V$.

It follows that M is a base of V over IF.