0. This handout is a continuation of the Handout *Partial orderings and total orderings*.

1. **Definition.**

Let A be a set, and T be a partial ordering in A with graph G. Write $u \preceq v$ exactly when $(u, v) \in G$.

Let B be a subset of A.

Let $\lambda \in B$. We say λ is a $\left\{ \begin{array}{c} \textbf{greatest} \\ \textbf{least} \end{array} \right\}$ **element** *of B with respect to T if, for any $x \in B$,* $\left\{ \begin{array}{c} x \preceq \lambda \\ x \succeq \lambda \end{array} \right\}$.

Remark. A subset of A has at most one greatest/least element with respect to T. Hence it makes sense to refer to such an element of A as 'the' greatest/least element with respect to T, if it exists.

Here in this Handout we focus on the question of existence of greatest/least elements for sets with respect to total orderings.

2. **Example (A'). (Usual ordering for real numbers.)**

The notion of greatest/least element for subsets of $\mathbb{R}$ with respect to the usual ordering for real numbers reduces to that for 'greatest/least element for subsets of $\mathbb{R}$', introduced in the Handout *Greatest/least element, upper/lower bound.*

(a) According to the Well-ordering Principle for Integers, for any subset B of $\mathbb{N}$, if B is non-empty, then B has a least element (with respect to the usual ordering for natural numbers).

A non-empty subset of $\mathbb{N}$ does not necessarily have any greatest element.

(b) Let a, b be real numbers. Supposed $a < b$.

	least element	greatest element
(a, b)	*nil*	*nil*
$[a, b)$	a	*nil*
$(a, b]$	*nil*	b
$[a, b]$	a	b

	least element	greatest element
$(a, +\infty)$	*nil*	*nil*
$[a, +\infty)$	a	*nil*
$(-\infty, b)$	*nil*	*nil*
$(-\infty, b]$	*nil*	b

3. **Definition.**

Let A be a set, and T be a partial ordering in A. We say T is a **well-order relation** *in A if the statement (λ) holds:*

> (λ) *For any subset B of A, if B is non-empty then B has a least element with respect to T.*

We also say that A is **well-ordered** *by T, and that the poset (A, T) is well-ordered.*

Simple examples and non-examples of well-ordered sets.

(a) $\mathbb{N}$ is well-ordered by the usual ordering for natural numbers, according to Example (A').
(This is just a re-formulation of the statement of the Well-ordering Principle for Integers.)
This is the primordial example of well-ordered sets.

(b) Every non-empty subset of $\mathbb{Z}$ which is bounded below in $\mathbb{Z}$ is well-ordered by the usual ordering for integers.
$\mathbb{Z}$ is not well-ordered by the usual ordering for integers. (Why?)

(c) $\mathbb{Q}$ is not well-ordered by the usual ordering for rational numbers. (Why?)

(d) $\mathbb{R}$ is not well-ordered by the usual ordering for real numbers. (Why?)

4. **Lemma (8).**

Let A be a set, and T is a partial ordering in A.

Suppose A is well-ordered by T. Then A is totally ordered by T.

Proof of Lemma (8).

Let A be a set, and T be a partial ordering in A with graph G. Suppose A is well-ordered by T.

Pick any $x, y \in A$. Define $B = \{x, y\}$.

Then B is a non-empty subset of A.

By assumption, A is well-ordered by T.

Then B has a least element with respect to T, say, x.

Therefore, by definition, $(x, y) \in G$. Therefore $(x, y) \in G$ or $(y, x) \in G$.

It follows that A is totally ordered by T.

Non-examples on well-order relations.

According to Lemma (8), there is no chance for a partial ordering which is not a total ordering to be a well-order relation.

- Refer to Example (B).

  The partial ordering T_{div} in $\mathbb{N}$ defined by divisibility is not a well-order relation, because it is not a total ordering in $\mathbb{N}$.

- Refer to Example (C).

  When E is a set which has at least two elements, $(\mathfrak{P}(E), \mathfrak{P}(E), G_{E,\text{subset}})$ is not a well-order relation, because it is not a total ordering in $\mathfrak{P}(E)$.

Reminder. The converse of Lemma (8) is false: a total ordering in a set is not necessarily a well-order relation in that set.

(For instance, the usual ordering for real numbers is a total ordering in $\mathbb{R}$ but it is not a well-order relation in $\mathbb{R}$.)

5. **Lemma (9).**

Let A be a set. Suppose T is a well-order relation in A with graph G.

Then, for any subset B of A, $(B, B, G \cap B^2)$ is a well-order relation in B.

6. **Theorem (10).**

Let A be a non-empty set. Suppose T is a well-order relation in A with graph G. Write $x \preceq y$ iff $(x, y) \in G$.

Then the statements below hold:

(a) *There exists some unique $\lambda \in A$ such that for any $x \in A \backslash \{\lambda\}$, $\lambda \prec x$.*

(b) *For any $x \in A$, if x is not a greatest element of A with respect to T then there exists some unique $y \in A$ such that $x \prec y$ and (for any $z \in A$, if $x \preceq z \preceq y$ then $z = x$ or $z = y$).*

Remark. Theorem (10) brings out what is special about well-ordered posets.

- Statement (a) says that some unique element of A, namely the least element of A with respect to T, will be the 'starting element' of A, in the sense that no element of A will precede it with respect to T.
- Statement (b) says that it makes sense to talk about the (unique) 'next element' of A for each element of A, in the sense that no third element of A will be between these two.

This allows us to visualize the 'ordering' of all the elements of A, with respect to T, in the 'chain of inequalities'

$$\lambda \preceq \lambda' \preceq \lambda'' \preceq \lambda''' \preceq \cdots$$

in which

λ is the least element of A with respect to T,
λ' is the least element of $A \backslash \{\lambda\}$ with respect to T,
λ'' is the least element of $A \backslash \{\lambda, \lambda'\}$ with respect to T,
λ''' is the least element of $A \backslash \{\lambda, \lambda', \lambda''\}$ with respect to T, et cetera.

An illustration is how we may visualize the 'ordering' for all natural numbers with respect to its usual ordering:

$$0 \leq 1 \leq 2 \leq 3 \leq 4 \leq \cdots$$

This cannot be done for the usual ordering for integers because $\mathbb{Z}$ has no least element.

This cannot be done for the usual ordering for rational numbers because the notion of 'next rational' number does not make sense: between any two distinct rational numbers there is definitely a third rational number.

But we may ask:

- *Is it possible to equip these sets with some other partial orderings which are well-order relations?*

7. **Example (D'). (Lexicographical ordering in $\mathbb{N}^2$ as a well-order relation in $\mathbb{N}^2$.)**

The lexicographical ordering in $\mathbb{N}^2$ is a well-order relation in $\mathbb{N}^2$ because the statement (†) holds:

(†) *For any subset B of $\mathbb{N}^2$, if B is non-empty, then B has a least element with respect to the lexicographical ordering in $\mathbb{N}^2$.*

Below is the idea for the argument for the statement (†). (The detail is left as an exercise.)

Suppose B is a non-empty subset of $\mathbb{N}^2$.

Then we may pick some element of B, say, the ordered pair of natural numbers, say, (u, v).

The lexicographical ordering in $\mathbb{N}^2$ allows us to visualize the 'ordering' for all the elements of $\mathbb{N}^2$, up to and including (u, v), through such a 'chain of inequalities' below:

$$(0,0) \leq_{\text{lex}} (0,1) \leq_{\text{lex}} (0,2) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (1,0) \leq_{\text{lex}} (1,1) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (2,0) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (u,0) \leq_{\text{lex}} (u,1) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (u,v)$$

So elements of B are listed in at least one of the rows $(\sharp_0)$, $(\sharp_1)$, $(\sharp_2)$, ..., $(\sharp_u)$, each with 'constant' first coordinate, in the table below:

$$\begin{array}{lcccccccccc}
(\sharp_0): & (0,0) & (0,1) & (0,2) & (0,3) & \cdots & (0,v-1) & (0,v) & (0,v+1) & \cdots \\
(\sharp_1): & (1,0) & (1,1) & (1,2) & (1,3) & \cdots & (1,v-1) & (1,v) & (1,v+1) & \cdots \\
(\sharp_2): & (2,0) & (2,1) & (2,2) & (2,3) & \cdots & (2,v-1) & (2,v) & (2,v+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
(\sharp_u): & (u,0) & (u,1) & (u,2) & (u,3) & \cdots & (u,v-1) & \boxed{(u,v)} & (u,v+1) & \cdots
\end{array}$$

← an element of B.

The Well-ordering Principle for Integers guarantees that there will be a row in this table with the 'smallest value of label', say, s, so that some element of B, say, (s,w) is listed in the row $(\sharp_s)$.

There is no element of B within the rows (#₀), (#₁), ..., (#ₛ₋₁).

$$\begin{array}{lcccccccccc}
(\sharp_0): & (0,0) & (0,1) & (0,2) & (0,3) & \cdots & (0,w-1) & (0,w) & (0,w+1) & \cdots \\
(\sharp_1): & (1,0) & (1,1) & (1,2) & (1,3) & \cdots & (1,w-1) & (1,w) & (1,w+1) & \cdots \\
(\sharp_2): & (2,0) & (2,1) & (2,2) & (2,3) & \cdots & (2,w-1) & (2,w) & (2,w+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
(\sharp_s): & (s,0) & (s,1) & (s,2) & (s,3) & \cdots & (s,w-1) & \boxed{(s,w)} & (s,w+1) & \cdots
\end{array}$$

$s \leq u$

← an element of B.

Then the Well-ordering Principle for Integers further guarantees that amongst

$$(s,0),(s,1),(s,2),\cdots,(s,w-1),(s,w),$$

there will be an element of B with the 'smallest second coordinate', say, t.

$$\begin{array}{lcccccccccc}
(\sharp_0): & (0,0) & (0,1) & (0,2) & (0,3) & \cdots & (0,t-1) & (0,t) & (0,t+1) & \cdots \\
(\sharp_1): & (1,0) & (1,1) & (1,2) & (1,3) & \cdots & (1,t-1) & (1,t) & (1,t+1) & \cdots \\
(\sharp_2): & (2,0) & (2,1) & (2,2) & (2,3) & \cdots & (2,t-1) & (2,t) & (2,t+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots & \\
(\sharp_s): & (s,0) & (s,1) & (s,2) & (s,3) & \cdots & (s,t-1) & (s,t) & (s,t+1) & \cdots \\
\vdots & \vdots & \vdots & \vdots & \vdots & & \vdots & \vdots & \vdots &
\end{array}$$

There is no element of B within the rows (#₀), (#₁), ⋯, (#ₛ₋₁).

There is no element of B within (s,0), ⋯, (s,t−1).

an element of B

(s,t) will be the least element of B with respect to the lexicographical ordering in $\mathbb{N}^2$.

Example (D') is an illustration of the idea in Theorem (11).

8. **Theorem (11).**

Let A, B be sets. Suppose R is a well-order relation in A, and S is a well-order relation in B.

Then the lexicographical ordering in $A \times B$ induced by R and S is a well-order relation in $A \times B$.

9. **Example (E). (Well-order relation in $\mathbb{Z}$ arising from the usual ordering for natural numbers.)**

Recall that that $\mathbb{Z}$ is not well-ordered by the usual ordering for integers.

How, we may define a well-order relation in $\mathbb{Z}$ with the help of the usual ordering for natural numbers.

Define the function $f : \mathbb{Z} \longrightarrow \mathbb{N}$ by $f(x) = \begin{cases} 2x & \text{if } x \text{ is non-negative} \\ -2x-1 & \text{if } x \text{ is negative} \end{cases}$.

f is an injective function from $\mathbb{Z}$ to $\mathbb{N}$.

Let $G = \{(x, y) \mid x \in \mathbb{Z} \text{ and } y \in \mathbb{Z} \text{ and } f(x) \leq f(y)\}$, and $S = (\mathbb{Z}, \mathbb{Z}, G)$.

S is a well-order relation in $\mathbb{Z}$.

So we visualize the 'ordering' for all integers with respect to the well-order relation S, through the 'chain of inequalities' below:

$$0 \preceq_S -1 \preceq_S 1 \preceq_S -2 \preceq_S 2 \preceq_S -3 \preceq_S 3 \preceq_S \cdots \preceq_S n-1 \preceq_S -n \preceq_S n \preceq_S \cdots$$

This is simply a direct translation, via f and S, of the chain of inequalities

$$0 \leq 1 \leq 2 \leq 3 \leq 4 \leq 5 \leq 6 \leq \cdots \leq 2n-2 \leq 2n-1 \leq 2n \leq \cdots .$$

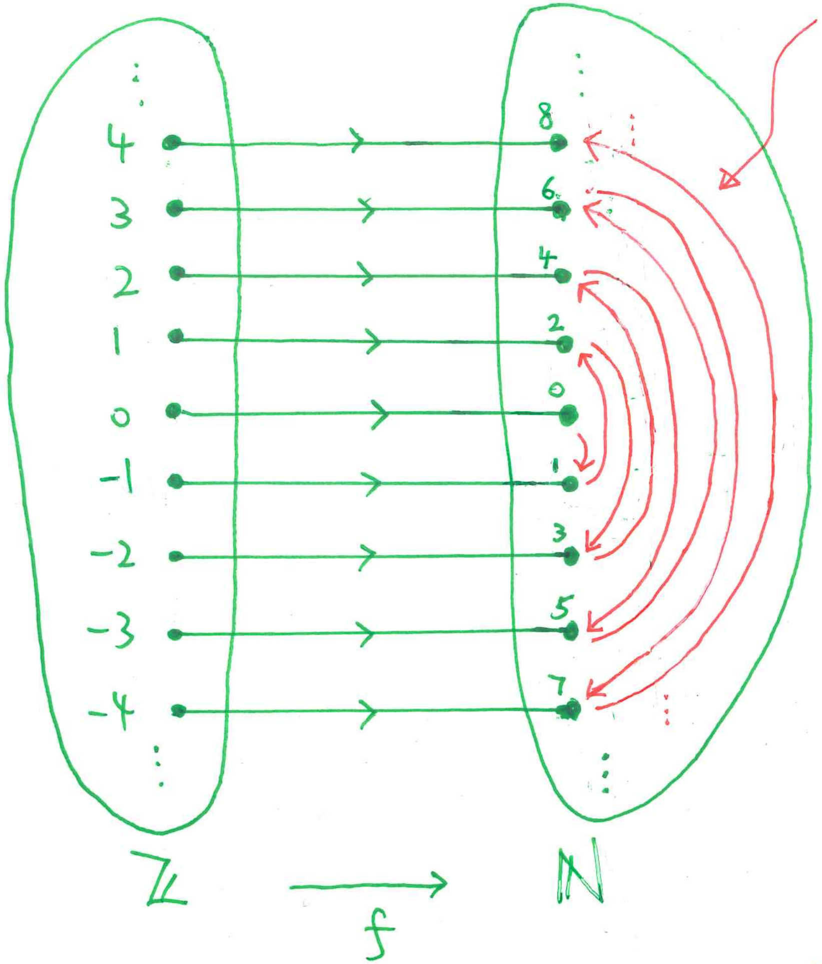
Note that $0 \prec_S -1 \prec_S 1$ whereas $-1 < 0 < 1$. Hence S is certainly distinct from the usual ordering for integers.

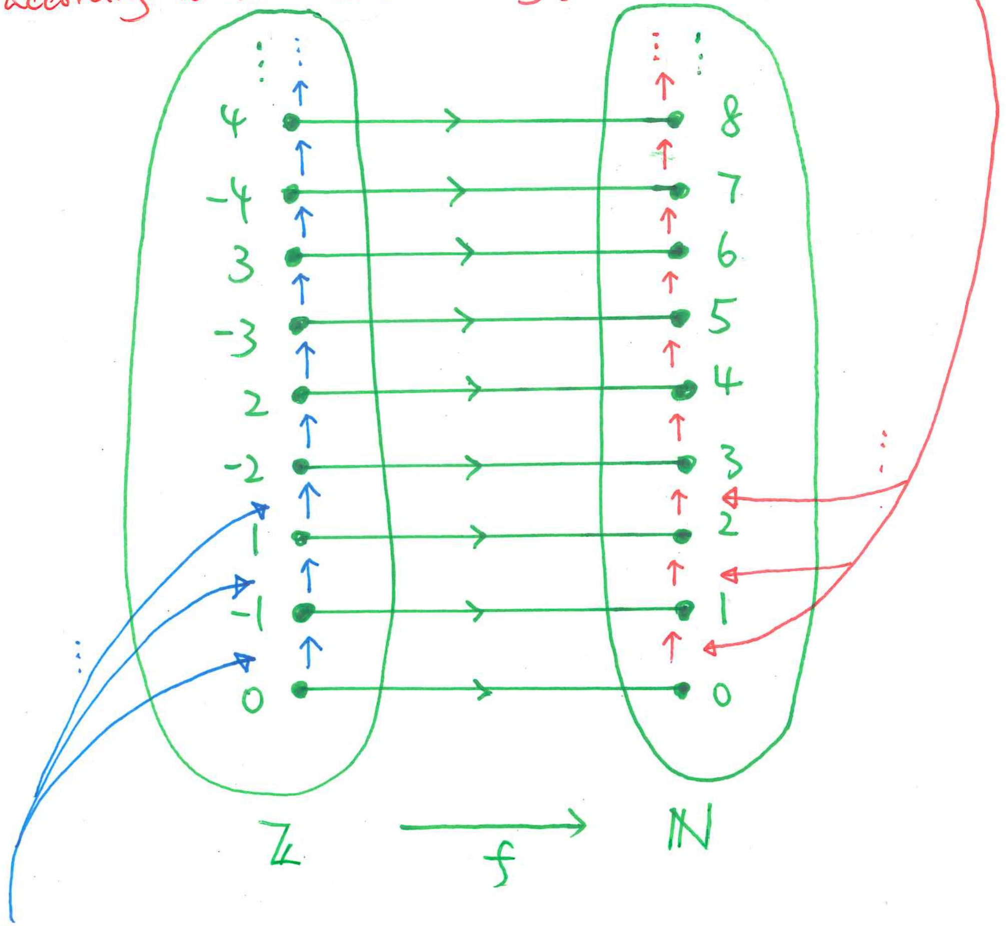
Visualization for Example (E).

The injective function $f: \mathbb{Z} \to \mathbb{N}$ is given by

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is non-negative.} \\ -2x-1 & \text{if } x \text{ is negative.} \end{cases}$$

This gives the ordering of natural numbers by succession according to the usual ordering for natural numbers.





This is the partial ordering S in $\mathbb{Z}$ defined by the pullback of the usual ordering for natural numbers by the injective function f. It is a well-order relation in $\mathbb{Z}$ because the usual ordering for natural numbers is a well-order relation in $\mathbb{N}$.

10. **Example (F). (Well-order relation in $\mathbb{N}^2$ which is not the same as the lexicographical ordering.)**

Recall that the lexicographical ordering in $\mathbb{N}^2$ is a well-order relation in $\mathbb{N}^2$. We now introduce, via an injective function from $\mathbb{N}^2$ to $\mathbb{N}$, another well-order relation in $\mathbb{N}^2$ which is not the lexicographical ordering in $\mathbb{N}^2$.

Define the function $f : \mathbb{N}^2 \longrightarrow \mathbb{N}$ by $f(x, y) = 2^x 3^y$ for any $x, y \in \mathbb{N}$.

f is an injective function from $\mathbb{N}^2$ to $\mathbb{N}$. (You need Euclid's Lemma to justify this claim.)

Let $G = \{((s,t),(u,v)) \mid s,t,u,v \in \mathbb{N} \text{ and } f(s,t) \leq f(u,v)\}$, and $S = (\mathbb{N}^2, \mathbb{N}^2, G)$.

S is a well-order relation in $\mathbb{N}^2$.

So we visualize the 'ordering' for all the elements of $\mathbb{N}^2$ with respect to the well-order relation S, through the 'chain of inequalities' below:

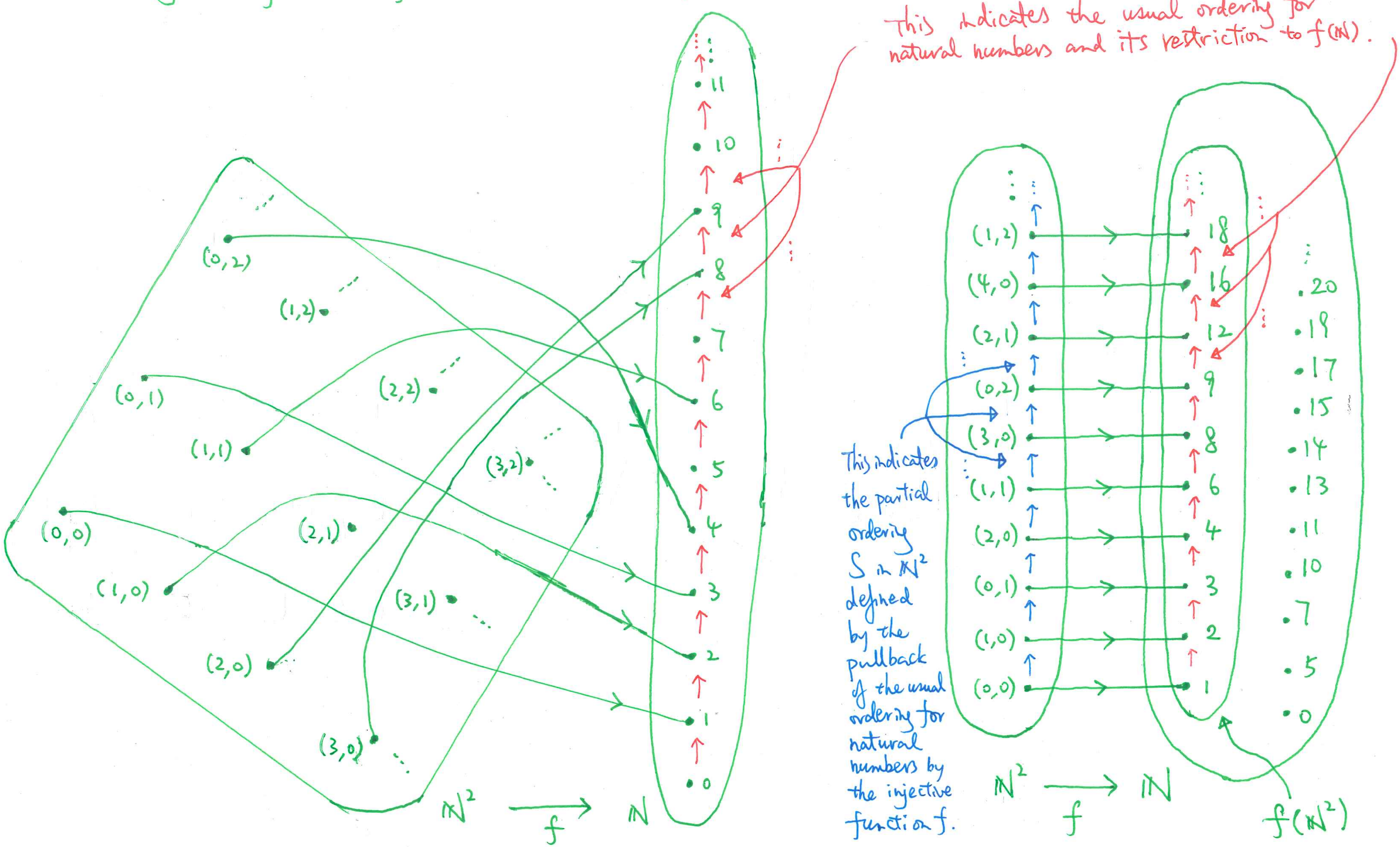
$$(0,0) \preceq_S (1,0) \preceq_S (0,1) \preceq_S (2,0) \preceq_S (1,1) \preceq_S (3,0) \preceq_S (0,2) \preceq_S (2,1) \preceq_S (4,0) \preceq_S (1,2) \preceq_S (3,1) \preceq_S (0,3) \preceq_S \cdots$$

This is simply a direct translation, via f and S, of the chain of inequalities

$$1 \leq 2 \leq 3 \leq 4 \leq 6 \leq 8 \leq 9 \leq 12 \leq 16 \leq 18 \leq 24 \leq 27 \leq \cdots.$$

Visualization for Example (F).

The injective function $f: \mathbb{N}^2 \to \mathbb{N}$ is given by $f(x,y) = 2^x \cdot 3^y$ for any $x, y \in \mathbb{N}$.



Note that

$$(1,0) \prec_S (0,1) \prec_S (2,0)$$

whereas

$$(1,0) <_{\text{lex}} (2,0) <_{\text{lex}} (0,1).$$

Hence S is certainly distinct from the lexicographical ordering for $\mathbb{N}^2$.

Remark. Replacing f by another injective function from $\mathbb{N}^2$ to $\mathbb{N}$, we will obtain another well-order relation in $\mathbb{N}^2$ from such a construction.

(For instance, what do you obtain with the injective function $g : \mathbb{N}^2 \longrightarrow \mathbb{N}$ given by $g(x,y) = 2^x 5^y$ for any $x, y \in \mathbb{N}$? Or how about the injective function $h : \mathbb{N}^2 \longrightarrow \mathbb{N}$ given by $h(x,y) = 3^x 5^y$ for any $x, y \in \mathbb{N}$?)

Example (E), Example (F) are illustrations of the idea in Theorem (12), which is concerned with general partial orderings.

11. **Theorem (12).**

Let A, B *be sets, and* $f : A \longrightarrow B$ *be an injective function.*

Suppose T *is a partial ordering in* B *with graph* H*. Write* $u \preceq_T v$ *exactly when* $(u, v) \in H$*.*

Define $G = \{(x, y) \mid x, y \in A \text{ and } f(x) \preceq_T f(y)\}$*, and* $S = (A, A, G)$*.*

Then S *is a partial ordering in* A *with graph* G*.*

If T *is a total ordering in* B *then* S *is a total ordering in* A*.*

If T *is a well-order relation in* B *then* S *is a well-order relation in* A*.*

Remark on terminology and notation. In the context of Theorem (12), the partial ordering S defined by the injective function f and the partial ordering T is called the **partial ordering in** A **defined by the pullback of** T **by** f. It is denoted by f^*T, and its graph is denoted by f^*H.

12. **Example (G). (Well-order relation in $\mathbb{Q}$ arising from the usual ordering for natural numbers.)**

Recall that that $\mathbb{Q}$ is not well-ordered by the usual ordering for integers.

However, we may define a well-order relation in $\mathbb{Q}$ with the help of the usual ordering for natural numbers.

(a) Refer to Example (E). We have constructed a well-order relation in $\mathbb{Z}$, namely, S, (with the help of the usual ordering for natural numbers).

(b) By Theorem (11), $\mathbb{Z}^2$ is well-ordered by the lexicographical ordering in $\mathbb{Z}^2$ induced by S and S.

We denote this well-order relation in $\mathbb{Z}^2$ by T.

S gives:

$$0 \preceq_S -1 \preceq_S 1 \preceq_S -2 \preceq_S 2 \preceq_S -3 \preceq_S 3 \preceq_S \cdots$$

T gives:

$$(0,0) \preceq_T (0,-1) \preceq_T (0,1) \preceq_T (0,-2) \preceq_T (0,2) \preceq_T (0,-3) \preceq_T (0,3) \preceq_T \cdots$$

$$\preceq_T (-1,0) \preceq_T (-1,-1) \preceq_T (-1,1) \preceq_T (-1,-2) \preceq_T (-1,2) \preceq_T (-1,-3) \preceq_T (-1,3) \preceq_T \cdots$$

$$\preceq_T (1,0) \preceq_T (1,-1) \preceq_T (1,1) \preceq_T (1,-2) \preceq_T (1,2) \preceq_T (1,-3) \preceq_T (1,3) \preceq_T \cdots$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

(c) We take the statement ($\sharp$) for granted:

($\sharp$) *For any* $r \in \mathbb{Q} \backslash \{0\}$*, there exist some unique* $p_r, q_r \in \mathbb{Z}$ *such that* $\gcd(p_r, q_r) = 1$ *and* $q_r > 0$ *and* $r = \dfrac{p_r}{q_r}$.

(Justify the statement ($\sharp$) as an exercise.)

Define the function $f : \mathbb{Q} \longrightarrow \mathbb{Z}^2$ by

$$f(r) = \begin{cases} (p_r, q_r) & \text{if} \quad r \in \mathbb{Q} \backslash \{0\} \\ (0, 1) & \text{if} \quad r = 0. \end{cases}$$

f is injective.

(d) According to Theorem (12), the partial ordering f^*T in $\mathbb{Q}$ defined by the pullback of T by f is a well-order relation in $\mathbb{Q}$.

How to visualize f^*T?

For any $u, v \in \mathbb{Q}$, write $u \preceq v$ exactly when (u,v) belongs to the graph of f^*T (i.e. $f(u) \preceq_T f(v)$).

$$0 = \frac{0}{1}$$

$$\preceq -1 = \frac{-1}{1} \preceq -\frac{1}{2} = \frac{-1}{2} \preceq -\frac{1}{3} = \frac{-1}{3} \preceq -\frac{1}{4} = \frac{-1}{4} \preceq -\frac{1}{5} = \frac{-1}{5} \preceq -\frac{1}{6} = \frac{-1}{6} \preceq -\frac{1}{7} = \frac{-1}{7} \preceq \cdots\cdots$$

$$\preceq 1 = \frac{1}{1} \preceq \frac{1}{2} \preceq \frac{1}{3} \preceq \frac{1}{4} \preceq \frac{1}{5} \preceq \frac{1}{6} \preceq \frac{1}{7} \preceq \cdots\cdots$$

$$\preceq -2 = \frac{-2}{1} \preceq -\frac{2}{3} = \frac{-2}{3} \preceq -\frac{2}{5} = \frac{-2}{5} \preceq -\frac{2}{7} = \frac{-2}{7} \preceq \cdots\cdots$$

$$\preceq 2 = \frac{2}{1} \preceq \frac{2}{3} \preceq \frac{2}{5} \preceq \frac{2}{7} \preceq \cdots\cdots$$

$$\preceq -3 = \frac{-3}{1} \preceq -\frac{3}{2} = \frac{-3}{2} \preceq -\frac{3}{4} = \frac{-3}{4} \preceq -\frac{3}{5} = \frac{-3}{5} \preceq -\frac{3}{7} = \frac{-3}{7} \preceq \cdots\cdots$$

$$\preceq 3 = \frac{3}{1} \preceq \frac{3}{2} \preceq \frac{3}{4} \preceq \frac{3}{5} \preceq \frac{3}{7} \preceq \cdots\cdots$$

$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Reminder. f^*T is not the usual ordering for rational numbers.

13. **Well-ordering Principle, as a fundamental assumption in mathematics.**

Example (E) and Example (G) tell us:

- Despite the fact that $\mathbb{Z}, \mathbb{Q}$ themselves are not well-ordered by the usual ordering for real numbers, it is still possible to equip them with various well-order relations.

We may ask: *Can we do the same thing for* $\mathbb{R}$*?*

If $\mathbb{R}$ can be equipped with a well-order relation, say, T, then the lexicographical ordering in $\mathbb{R}^2$ induced by T will be a well-order relation in $\mathbb{R}^2$, and will further provide a well-order relation for $\mathbb{C}$.

We may further ask: *Is it possible to equip any arbitrary set equipped with a well-order relation?*

It turns out that the answers to these questions are not quite trivial.

Well-ordering Principle.

Suppose A is a set. Then there exists some partial ordering T in A such that A is well-ordered by T.

Remark. We do not 'prove' the Well-ordering Principle. It is taken as a fundamental assumption in mathematics. (Of course, it is legitimate to choose between '*believing*' the Well-ordering Principle and '*not believing*' it.)