1. Compare Theorem (1), Theorem (2), Theorem (3):

Theorem (1).

Let B be a subset of $\mathbb{R}$. The statements below hold:

(a) *For any $x \in B$, $x \leq x$.*

(b) *For any $x, y \in B$, if $x \leq y$ and $y \leq x$ then $x = y$.*

(c) *For any $x, y, z \in B$, if $x \leq y$ and $y \leq z$ then $x \leq z$.*

Theorem (2).

The statements below hold:

(a) *Suppose $x \in \mathbb{N}$. Then x is divisible by x.*

(b) *Let $x, y \in \mathbb{N}$. Suppose y is divisible by x and x is divisible by y. Then $x = y$.*

(c) *Let $x, y, z \in \mathbb{N}$. Suppose y is divisible by x and z is divisible by y. Then z is divisible by x.*

Suppose we write $u \mid v$ exactly when v is divisible by u.
Then (a), (b), (c) become:

(a) For any $x \in \mathbb{N}$, $x \mid x$.
(b) For any $x, y \in \mathbb{N}$, if $x \mid y$ and $y \mid x$ then $x = y$.
(c) For any $x, y, z \in \mathbb{N}$, if $x \mid y$ and $y \mid z$ then $x \mid z$.

Theorem (1).

Let B be a subset of $\mathbb{R}$. The statements below hold:

(a) *For any $x \in B$, $x \leq x$.*

(b) *For any $x, y \in B$, if $x \leq y$ and $y \leq x$ then $x = y$.*

(c) *For any $x, y, z \in B$, if $x \leq y$ and $y \leq z$ then $x \leq z$.*

Theorem (3).

Let E be a set. The statements below hold:

(a) *For any $A \in \mathfrak{P}(E)$, $A \subset A$.*

(b) *For any $A, B \in \mathfrak{P}(E)$, if $A \subset B$ and $B \subset A$ then $A = B$.*

(c) *For any $A, B, C \in \mathfrak{P}(E)$, if $A \subset B$ and $B \subset C$ then $A \subset C$.*

Theorem (1), Theorem (2), Theorem (3) suggest the presence of some common structure for various mathematical objects. This mathematical structure is usually referred to as **partial ordering**.

2. **Definition.**

Let A be a set, and T be a relation in A with graph G.

(a) *T is said to be* **reflexive** *if the statement (ρ) holds:*

(ρ): *For any $x \in A$, $(x, x) \in G$.*

(b) *T is said to be* **anti-symmetric** *if the statement (α) holds:*

(α): *For any $x, y \in A$, if $((x, y) \in G$ and $(y, x) \in G)$ then $x = y$.*

(c) *T is said to be* **transitive** *if the statement (τ) holds:*

(τ): *For any $x, y, z \in A$, if $((x, y) \in G$ and $(y, z) \in G)$ then $(x, z) \in G$.*

Remark. The notions of reflexivity, anti-symmetry, and transitivity are 'logically independent' of each other.

3. **Definition.**

Let A be a set, and T be a relation in A with graph G.

T is said to be a **partially ordering** *in A if T is reflexive, anti-symmetric and transitive.*

We may also say that A is partially ordered by T. We may refer to the ordered pair (A, T) as a **poset**.

4. **Example (A). (Usual ordering for real numbers.)**

Theorem (1), which is concerned with the usual ordering for real numbers, can be re-formulated as:

Suppose B is a subset of $\mathbb{R}$. Define $G = \{(x, y) \mid x, y \in B$ and $x \leq y\}$. Then (B, B, G) is a partial ordering.

How?

By the definition of G, for any $x, y \in B$, $(x, y) \in G$ iff $x \leq y$.

So, in terms of G, the statement of Theorem (1) becomes:

Let B be a subset of $\mathbb{R}$. The statements below hold:

(a) For any $x \in B$, $\underbrace{(x, x) \in G}_{x \leq x}$.

(b) For any $x, y \in B$, if $\underbrace{(x, y) \in G}_{x \leq y}$ and $\underbrace{(y, x) \in G}_{y \leq x}$ then $x = y$.

(c) For any $x, y, z \in B$, if $\underbrace{(x, y) \in G}_{x \leq y}$ and $\underbrace{(y, z) \in G}_{y \leq z}$ then $\underbrace{(x, z) \in G}_{x \leq z}$.

Example (A). (Usual ordering for real numbers.)

Theorem (1), which is concerned with the usual ordering for real numbers, can be reformulated as:

Suppose B is a subset of $\mathbb{R}$. Define $G = \{(x, y) \mid x, y \in B \text{ and } x \leq y\}$.
Then (B, B, G) is a partial ordering.

Remark. Example (A) is the primordial example of partial orderings. The notations and terminologies for general partial orderings, soon to be introduced, are inspired by the usual ordering for real numbers.

We may think of the usual orderings in $\mathbb{N}, \mathbb{Z}, \mathbb{Q}$ as 'restrictions' to these sets of the usual ordering for real numbers.

5. **Lemma (4).**

Let A be a set. Suppose T is a partial ordering in A with graph G.
Then, for any subset C of A, $(C, C, G \cap C^2)$ is a partial ordering in C.

Remarks on terminologies and notations.

We call the partial ordering $(C, C, G \cap C^2)$ the **restriction** of T to C.

6. **Example (B). (Divisibility for natural numbers.)**

Theorem (2), which is concerned with divisibility for natural numbers, can be re-formulated as:

> *Define* $G_{\text{div}} = \{(x, y) \mid x, y \in \mathbb{N}$ *and* y *is divisible by* $x\}$, *and* $T_{\text{div}} = (\mathbb{N}, \mathbb{N}, G_{\text{div}})$.
> *Then* T_{div} *is a partial ordering in* $\mathbb{N}$.

We call T_{div} the partial ordering in $\mathbb{N}$ defined by divisibility.

Remark. By Lemma (4), the restriction of T_{div} to any subset B of $\mathbb{N}$ defines a partial ordering in B.

Example. Suppose $B = [\![0, 9]\!]$. Then $(B, B, G_{\text{div}} \cap B^2)$ is a partial ordering, with

$$
\begin{aligned}
G_{\text{div}} \cap B^2 = \{ & (0,0), \\
& (1,0), (1,1), (1,2), (1,3), (1,4), (1,5), (1,6), (1,7), (1,8), (1,9), \\
& (2,0), (2,2), (2,4), (2,6), (2,8), \\
& (3,0), (3,3), (3,6), (3,9), \\
& (4,0), (4,4), (4,8), \\
& (5,0), (5,5), \\
& (6,0), (6,6), \\
& (7,0), (7,7), \\
& (8,0), (8,8), \\
& (9,0), (9,9) \}.
\end{aligned}
$$

Example (B). (Divisibility for natural numbers.)

Theorem (2), which is concerned with divisibility for natural numbers, can be re-formulated as:

> *Define* $G_{\text{div}} = \{(x, y) \mid x, y \in \mathbb{N}$ *and* y *is divisible by* $x\}$, *and* $T_{\text{div}} = (\mathbb{N}, \mathbb{N}, G_{\text{div}})$.
> *Then* T_{div} *is a partial ordering in* $\mathbb{N}$.

We call T_{div} the partial ordering in $\mathbb{N}$ defined by divisibility.

Remark. By Lemma (4), the restriction of T_{div} to any subset B of $\mathbb{N}$ defines a partial ordering in B.

Further remark. Although the usual ordering for natural numbers and T_{div} are both partial orderings in $\mathbb{N}$, there is a subtle but important difference between them:

- Every pair of natural numbers can be 'compared' in terms of the usual ordering. This is more formally formulated as:

  For any $x, y \in \mathbb{N}$, $x \leq y$ *or* $y \leq x$.

- Not every pair of natural numbers can be 'compared' in terms of T_{div}. This is more formally formulated as:

  There exists some $x, y \in \mathbb{N}$ *such that* $(x, y) \notin G_{\text{div}}$ *and* $(y, x) \notin G_{\text{div}}$.

  For instance, $(2, 3) \notin G_{\text{div}}$ and $(3, 2) \notin G_{\text{div}}$.

7. **Definition.**

Let A be a set, and T be a partial ordering in A with graph G.

(a) *Let $x, y \in A$. We say that x, y are* T**-comparable** *if $(x, y) \in G$ or $(y, x) \in G$.*

(b) We say that T is **strongly connected** (or **connex**) if the statement (κ) holds:

(κ): *For any $x, y \in A$, $(x, y) \in G$ or $(y, x) \in G$.*

8. **Definition.**

Let A be a set, and T be a partial ordering in A with graph G.

(a) *T is called a* **total ordering** *in A if T is strongly connected.*

We may also say that A is totally ordered by T, and that the poset (A, T) is totally ordered.

(b) *Let C be a subset of A.*

The set C is called a **chain** *with respect to T if the restriction of T to C is a total ordering in C.*

9. **Lemma (5).**

Suppose A be a set, and T is a partial ordering in A. Then the statements below are logically equivalent:

(a) *T is strongly connected.*

(b) *T is a total ordering in A.*

(c) *A is a chain with respect to T.*

Moreover, if T is a total ordering in A, then for any subset B of A, the restriction of T to B is a total ordering in B.

10. **Examples and non-examples on total orderings and chains.**

(a) Refer to Example (A).

For any subset B of $\mathbb{R}$, the usual ordering in B defines a total ordering in B.

(b) Refer to Example (B).

The partial ordering T_{div} in $\mathbb{N}$ defined by divisibility is not a total ordering in $\mathbb{N}$.

There are many proper subsets of $\mathbb{N}$ which are chains with respect to T_{div}, for instance:

$$\{0\}, \qquad \{2^k \mid k \in \mathbb{N}\}, \qquad \{3^k \mid k \in \mathbb{N}\}.$$

However, none of the sets below is a chain with respect to T_{div}:

$$\{2^j \mid j \in \mathbb{N}\} \cup \{3^k \mid k \in \mathbb{N}\}, \qquad \{2^j \cdot 3^k \mid j, k \in \mathbb{N}\}.$$

2, 3 belong to this set but they are not T_{div}-comparable.

2, 3 belong to this set but they are not T_{div}-comparable.

11. **Example (C). (Subset relation.)**

Theorem (3), which is concerned with the subset relation within an arbitrarily given set, can be re-formulated as:

> *Suppose E is a set. Define $G_{E,\text{subset}} = \{(U, V) \mid U, V \in \mathfrak{P}(E) \text{ and } U \subset V\}$, and $T_{E,\text{subset}} = (\mathfrak{P}(E), \mathfrak{P}(E), G_{E,\text{subset}})$.*
> *Then $T_{E,\text{subset}}$ is a partial ordering in $\mathfrak{P}(E)$.*

We call $T_{E,\text{subset}}$ the partial ordering in $\mathfrak{P}(E)$ defined by the subset relation.

When E contains two or more elements, $T_{E,\text{subset}}$ is not a total ordering.

By Lemma (4), the restriction of the partial ordering $T_{E,\text{subset}}$ to any subset of $\mathfrak{P}(E)$ defines a partial ordering on that subset of $\mathfrak{P}(E)$.

Example. Suppose $E = \{0, 1, 2\}$. Then $\mathfrak{P}(E) = \{\emptyset, \{0\}, \{1\}, \{2\}, \{1,2\}, \{0,2\}, \{0,1\}, E\}$.
Write $a = \{0\}$, $b = \{1\}$, $c = \{2\}$, $a' = \{1,2\}$, $b' = \{0,2\}$, $c' = \{0,1\}$.
The graph $G_{E,\text{subset}}$ of the partial ordering $T_{E,\text{subset}}$ is given by

$G_{E,\text{subset}} = \{(\emptyset,\emptyset), (\emptyset,a), (\emptyset,b), (\emptyset,c), (\emptyset,a'), (\emptyset,b'), (\emptyset,c'), (\emptyset,E),$
$(a,a), (a,b'), (a,c'), (a,E), (b,b), (b,a'), (b,c'), (b,E),$
$(c,c), (c,a'), (c,b'), (c,E), (a',a'), (a',E), (b',b'), (b',E), (c',c'), (c',E),$
$(E,E)\}$.

12. **Conventions on notations for partial orderings.**

We are going to introduce some conventions on notations for general partial orderings. They are inspired by the notations for usual orderings for real numbers and those for subset relations.

Let A be a set and T be a partial ordering in A with graph G.

Suppose we agree to write $(x, y) \in G$ as $x \preceq_T y$.

We pronounce '$x \preceq_T y$' as

'*x precedes or equals y under the partial ordering T*'.

(a) If T is the only partial ordering in A under consideration, we may drop the reference to the symbols T, G and write:

- '$x \preceq y$' in place of ' $x \preceq_T y$;
- '*A is partially ordered by $\preceq$*' in place of *A is partially ordered by T*;
- '*$(A, \preceq)$ is a poset*' in place of *(A, T) is a poset*; et cetera.

Under the above conventions, the statements (ρ), (α), (τ) that hold for the partial ordering T are re-formulated as:

(ρ): *For any $x \in A$, $x \preceq x$.*

(α): *For any $x, y \in A$, if ($x \preceq y$ and $y \preceq x$) then $x = y$.*

(τ): *For any $x, y, z \in A$, if ($x \preceq y$ and $y \preceq z$) then $x \preceq z$.*

(b) We also agree that the same symbol $\preceq$ will be used for the restriction of T to any subset of A.

(c) We may write '$x \preceq_T y$' as '$y \succeq_T x$'. The latter is pronounced as

'*y succeeds or equals x under the partial ordering T*'

(d) We may write $x \prec_T y$, or equivalently, $y \succ_T x$, exactly when $(x, y) \in G$ and $x \neq y$.

We pronounce '$x \prec_T y$' as

'*x precedes y under the partial ordering T*'.

We pronounce '$y \succ_T x$' as

'*y succeeds x under the partial ordering T*'.

Warning. Care must be taken because of the visual resemblance between the symbol $\leq$ and the symbol $\preceq$.

When we are using the symbol '$\preceq$' for formulating statements concerned with a general partial ordering T in an arbitrary set A, we have to deliberately remind ourselves that the statement (κ) may fail to hold:

(κ): *For any $x, y \in A$, $x \preceq y$ or $y \preceq x$.*

In fact (κ) holds exactly when the partial ordering T is a total ordering in A.

13. **Lemma (6).**

Let A be a set, and T be a partial ordering in A with graph G. Write $u \preceq v$ exactly when $(u, v) \in G$.

(a) *Let $x, y \in A$. The statements below are logically equivalent:*

i. x, y are T-comparable. ($x \preceq y$ or $y \preceq x$.)

ii. *Exactly one of '$x \prec y$', '$x = y$', '$x \succ y$' is true.*

(b) *T is strongly connected iff the statement $(\tau\chi)$ holds:*

$(\tau\chi)$: *For any $x, y \in A$, exactly one of '$x \prec y$', '$x = y$', '$x \succ y$' is true.*

Remark. When T is indeed a total ordering in A, the statement $(\tau\chi)$ is known as the **Law of Trichotomy** in the poset (A, T).

Proof of (a).

Let $x, y \in A$.

- [(ii) ⇒ (i) ?]

Suppose exactly one of '$x \prec y$', '$x = y$', '$x \succ y$' holds. Then by definition of $\preceq$, we have $x \preceq y$ or $y \preceq x$.

- [(i) ⇒ (ii) ?]

Suppose $x \preceq y$ or $y \preceq x$. Note that $x = y$ or $x \neq y$.

(Case 1). Suppose $x = y$. Then '$x \prec y$' is false. So is '$x \succ y$'.

(Case 2). Suppose $x \neq y$. Then $x \prec y$ or $y \prec x$.

(Case 2a). Suppose $x \prec y$. Then '$y \prec x$' is false; otherwise by anti-symmetry $x = y$.

(Case 2b). Suppose $y \prec x$. Then '$x \prec y$' is false; otherwise by anti-symmetry $x = y$.

14. **Example (D). (Lexicographical ordering in $\mathbb{N}^2$.)**

With the usual ordering in $\mathbb{N}$, we are going to construct a total ordering in $\mathbb{N}^2$, which is inspired by how words in a dictionary are arranged according to alphabetical order.

Define $J = \left\{ ((s,t),(u,v)) \;\middle|\; \begin{array}{l} s,t,u,v \in \mathbb{N}, \text{ and} \\ [s < u \text{ or } (s = u \text{ and } t \leq v)] \end{array} \right\}$, and $T = (\mathbb{N}^2, \mathbb{N}^2, J)$. Note that $J \subset (\mathbb{N}^2)^2$.

With a straightforward calculation, we can verify that T is a total ordering in $\mathbb{N}^2$.

The total ordering T is called the **lexicographical ordering** in $\mathbb{N}^2$.

For any $s,t,u,v \in \mathbb{N}$, we write $(s,t) \leq_{\text{lex}} (u,v)$ exactly when $((s,t),(u,v)) \in J$.

Then by definition, $(s,t) \leq_{\text{lex}} (u,v)$ iff $[s < u \text{ or } (s = u \text{ and } t \leq v)]$.

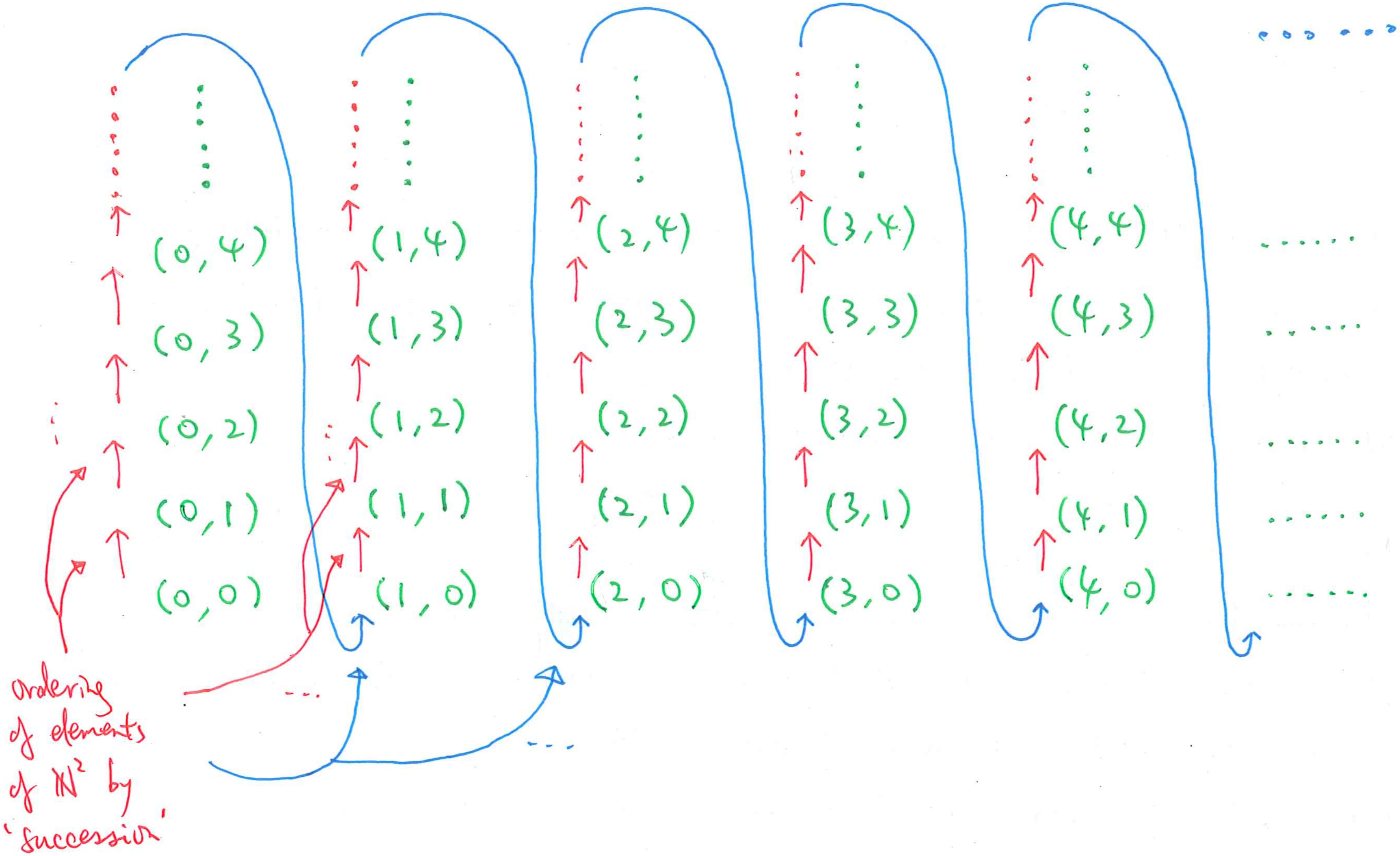
Illustrations:

- $(1,3) <_{\text{lex}} (2,0)$. (Reason: $1 < 2$.)
- $(2,3) <_{\text{lex}} (2,4)$. (Reason: $2 = 2$ and $3 < 4$.)

As a whole, T can be visualized as:

$$(0,0) \leq_{\text{lex}} (0,1) \leq_{\text{lex}} (0,2) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (1,0) \leq_{\text{lex}} (1,1) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (2,0) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (3,0) \leq_{\text{lex}} \cdots \leq_{\text{lex}} (4,0) \leq_{\text{lex}} \cdots$$

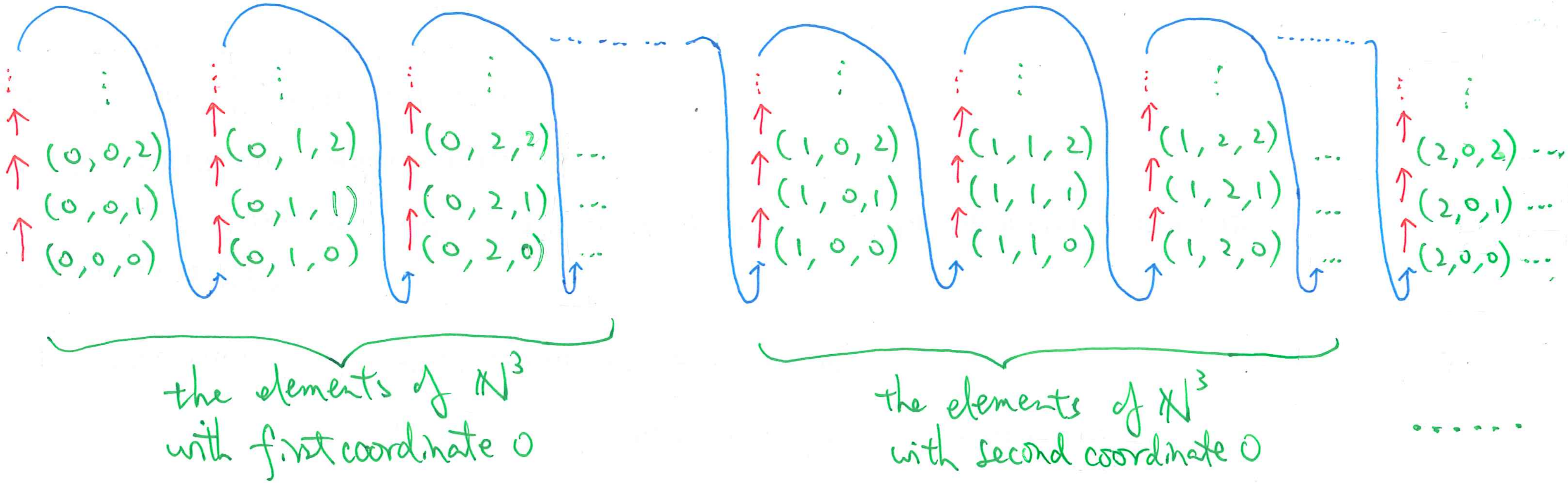
Visualization of the lexicographical ordering in $\mathbb{N}^2$.



Visualization of the lexicographical ordering in $\mathbb{N}^3$.

The graph J' of such a total ordering is given by

$$J' = \{((r,s,t),(u,v,w)) \mid r,s,t,u,v,w \in \mathbb{N}, \text{ and } [r<u \text{ or } (r=u \text{ and } s<v) \text{ or } (r=u \text{ and } s=v \text{ and } t \leq w)]\}.$$



Remark. We can apply the same method to construct the lexicographical ordering $\leq_{\text{lex}}$ in $\mathbb{N}^3$, given by $(\star_3)$:

$(\star)$ For any $r, s, t, u, v, w \in \mathbb{N}$, $(r, s, t) \leq_{\text{lex}} (u, v, w)$ iff $[r < u$ or $(r = u$ and $s < v)$ or $(r = u$ and $s = v$ and $t \leq w)]$.

We can 'inductively' construct the lexicographical ordering in $\mathbb{N}^k$ for each $k \in \mathbb{N} \backslash \{0\}$.

Further remark. The above constructions ultimately rely on the fact that the usual ordering in $\mathbb{N}$ is a total ordering in $\mathbb{N}$. No other aspect of the natural number system has anything to do with this construction.

Imitating the above construction, We can construct the lexicographical ordering $\leq_{\text{lex}}$ in $\mathbb{R}^2$ from the usual ordering in $\mathbb{R}$, which is given by $(\star)$:

$(\star)$ For any $s, t, u, v \in \mathbb{R}$, $(s, t) \leq_{\text{lex}} (u, v)$ iff $[s < u$ or $(s = u$ and $t \leq v)]$.

The lexicographical ordering in $\mathbb{N}^2$ is the restriction of this total ordering in $\mathbb{R}^2$.

We can 'inductively' construct the lexicographical ordering in $\mathbb{R}^k$ for each $k \in \mathbb{N} \backslash \{0\}$.

Example (D) is an illustration of the idea in Theorem (7), which is concerned with general partial orderings.

15. **Theorem (7).**

Let A, B be sets. Suppose R is a partial ordering in A with graph G, and S is a partial ordering in B with graph H.

Write $s \preceq_R u$ exactly when $(s, u) \in G$. Write $t \preceq_S v$ exactly when $(t, v) \in H$.

Define $J = \left\{ ((s,t),(u,v)) \;\middle|\; \begin{array}{l} s, u \in A, \text{ and } \; t, v \in B, \text{ and} \\ [s \prec_R u \text{ or } \; (s = u \text{ and } \; t \preceq_S v)] \end{array} \right\}$, *and* $T = (A \times B, A \times B, J)$.

Then T is a partial ordering in $A \times B$.

Moreover, if R is a total ordering in A and S is a total ordering in B, then T is a total ordering in $A \times B$.

Remark on terminologies and notations. T is called the **lexicographical ordering** in $A \times B$ induced by R and S.