MATH1050 Examples: Theoretical results on functions.

1. Prove the statement (♯):

   (♯) *Let A, B, C be sets and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. For any subset S of A, $(g \circ f)(S) = g(f(S))$.*

2. (a) Prove the statement (♯):

   (♯) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subsets U, V of B, if $U \subset V$ then $f^{-1}(U) \subset f^{-1}(V)$.*

   (b) Dis-prove the statement (♭):

   (♭) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subsets U, V of B, if $f^{-1}(U) \subset f^{-1}(V)$ then $U \subset V$.*

   (c) *You may apply the result obtained in the first part of this question to help simplify your arguments in this part.*
   Prove the statements below:

   i. *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subsets U, V of B, $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.*

   ii. *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subsets U, V of B, $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.*

3. (a) Prove the statement (♯):

   (♯) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subsets S, T of A, $f(S \cap T) \subset f(S) \cap f(T)$.*

   (b) Dis-prove the statement (♭):

   (♭) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subsets S, T of A, $f(S) \cap f(T) \subset f(S \cap T)$.*

   (c) i. *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose that for any subsets S, T of A, $f(S) \cap f(T) \subset f(S \cap T)$. Then f is injective.*

   ii. *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose f is injective. Then for any subsets S, T of A, $f(S) \cap f(T) \subset f(S \cap T)$.*

4. (a) Prove the statement (♯):

   (♯) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subset S of A, $S \subset f^{-1}(f(S))$.*

   (b) Dis-prove the statement (♭):

   (♭) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subset S of A, $f^{-1}(f(S)) \subset S$.*

   (c) i. *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose that for any subset S of A, $f^{-1}(f(S)) \subset S$. Then f is injective.*

   ii. *Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose f is injective. Then for any subset S of A, $f^{-1}(f(S)) \subset S$.*

5. *You may apply results already established in the lectures or in other exercises to help simplify the argument here.*
   Let A, B be sets, and $f : A \longrightarrow B$ be a function. Prove the following statements:

   (a) *$f(S) = f(f^{-1}(f(S)))$ for any subset S of A.*

   (b) *$f^{-1}(U) = f^{-1}(f(f^{-1}(U)))$ for any subset U of B.*

6. Prove the statement (♯):

   (♯) *Let A, B be sets, and $f : A \longrightarrow B$ be a function. For any subset S of A, for any subset U of B, $f(S \cap f^{-1}(U)) = f(S) \cap U$.*

7. We introduce the definitions and notations below:

   Let A, B be sets and $f : A \longrightarrow B$ be a function.

   f induces the pair of functions $f_{\mathfrak{P}} : \mathfrak{P}(A) \longrightarrow \mathfrak{P}(B)$, $f^{\mathfrak{P}} : \mathfrak{P}(B) \longrightarrow \mathfrak{P}(A)$, given by $f_{\mathfrak{P}}(S) = f(S)$ for any $S \in \mathfrak{P}(A)$, $f^{\mathfrak{P}}(U) = f^{-1}(U)$ for any $U \in \mathfrak{P}(B)$ respectively.

   (a) Let $f : A \longrightarrow B$, $g : A \longrightarrow B$ be functions. Prove that the statements (A), (B), (C) are logically equivalent:

   (A) $f = g$.

   (B) $f_{\mathfrak{P}} = g_{\mathfrak{P}}$.

(C) $f^{\mathfrak{P}} = g^{\mathfrak{P}}$.

(b) Let $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Prove the statements below:

i. $(g \circ f)_{\mathfrak{P}} = g_{\mathfrak{P}} \circ f_{\mathfrak{P}}$.

ii. $(g \circ f)^{\mathfrak{P}} = f^{\mathfrak{P}} \circ g^{\mathfrak{P}}$.

(c) Let $f : A \longrightarrow B$ be a function. Prove that the statements (D), (E), (F) are logically equivalent:

(D) *f is injective.*

(E) *$f_{\mathfrak{P}}$ is injective.*

(F) *$f^{\mathfrak{P}}$ is surjective.*

8. Let A, B be sets, and $f, g : A \longrightarrow B$ be functions, with graphs F, G respectively. Prove that the statements (†), (‡) are equivalent to each other:

(†) *$f(x) = g(x)$ for any $x \in A$.*

(‡) *$F = G$.*

Remark. What is the point of this question? We have introduced two notions of 'equality of functions': equality of functions as ordered triples, and equality of functions as 'assignments from A to B'. Are they in conflict? By virtue of the equivalence of the statements (†), (‡), these two notions of 'equality of functions' are consistent.

9. Prove the statements below:

(a) *Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$, $h : B \longrightarrow A$ be functions. Supppose $g \circ f = \mathsf{id}_A$ and $f \circ h = \mathsf{id}_B$. Then f is bijective, and $f^{-1} = g = h$.*

(b) *Let A, B be sets, and $f : A \longrightarrow B$ be a bijective function. $(f^{-1})^{-1} = f$.*

(c) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be bijective functions. $g \circ f$ is a bijective function and $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.*

10. (a) Prove the statements below:

i. *Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose there exists some function $g : B \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$. Then f is injective.*

ii. *Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose f is injective. Then there exists some function $g : B \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$.*

Remark. To write down a simple argument, we need the notion of bijective function.

(b) Prove the statements below:

i. *Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose there exists some function $g : B \longrightarrow A$ such that $f \circ g = \mathsf{id}_B$. Then f is surjective.*

ii. *Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose f is surjective. Then there exists some function $g : B \longrightarrow A$ such that $f \circ g = \mathsf{id}_B$.*

Remark. Re-read your argument after finishing it. Do you believe you have indeed proved anything?

(c) Dis-prove the statements below:

i. *Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose there exists some function $g : B \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$. Then f is surjective.*

ii. *Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose there exists some function $g : B \longrightarrow A$ such that $f \circ g = \mathsf{id}_B$. Then f is injective.*

11. Dis-prove each of the statements below.

(a) *Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose f is injective. Then there exists some function $g : B \longrightarrow A$ such that $f \circ g = \mathsf{id}_B$.*

(b) *Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Suppose f is surjective. Then there exists some function $g : B \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$.*

(c) *Let A be a non-empty set, and $f : A \longrightarrow A$ be a function. Suppose there exists some function $g : A \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$. Then f is surjective.*

(d) *Let A be a non-empty set, and $f : A \longrightarrow A$ be a function. Suppose there exists some function $g : A \longrightarrow A$ such that $f \circ g = \mathsf{id}_A$. Then f is injective.*

(e) *Let A be a non-empty set, and $f : A \longrightarrow A$ be a function. Suppose f is injective. Then there exists some function $g : A \longrightarrow A$ such that $f \circ g = \mathsf{id}_A$.*

(f) *Let A be non-empty set, and $f : A \longrightarrow A$ be a function. Suppose f is surjective. Then there exists some function $g : A \longrightarrow A$ such that $g \circ f = \mathsf{id}_A$.*

12. Let A, B, F be sets. Suppose $f = (A, B, F)$ is a function. Define $\hat{f} = (A, f(A), F)$. Prove the statements below:

(a) $F \subset A \times f(A)$.

(b) *$\hat{f}$ is a surjective function.*

(c) *Suppose f is injective, and suppose $H = \{(f(y), y) \mid y \in A\}$. Then $\hat{f}$ is a bijective function, with the inverse function of $\hat{f}$ being given by $(f(A), A, H)$.*

13. We introduce/recall some definitions:

- *Let A, B be sets, and $g : A \longrightarrow B$ be a function. Suppose C is a subset of A. The function $g\big|_C : C \longrightarrow B$ defined by $g\big|_C(x) = g(x)$ for any $x \in A$ is called the* **restriction of g to C**.
- *Let A, B be sets. The function $\pi_A : A \times B \longrightarrow A$ defined by $\pi_A(x, y) = x$ for any $x \in A$, $y \in B$ is called the* **projection function from $A \times B$ to A**. *The function $\pi_B : A \times B \longrightarrow B$ defined by $\pi_B(x, y) = y$ for any $x \in A$, $y \in B$ is called the* **projection function from $A \times B$ to B**.

Let $f = (A, B, G)$ be a relation. Consider the respective projection functions $\pi_A : A \times B \longrightarrow A$, $\pi_B : A \times B \longrightarrow B$, and the respective restrictions $\pi_A\big|_G : G \longrightarrow A$, $\pi_B\big|_G : G \longrightarrow B$.

(a) Prove the statement (♯):

(♯) The relation f is a function iff the function $\pi_A\big|_G$ is bijective.

(b) Now suppose the relation f is a function. Prove the statements below:

i. $\pi_B\big|_G = f \circ \pi_A\big|_G$.

ii. *The function f is surjective iff the function $\pi_B\big|_G$ is surjective.*

iii. *The function f is injective iff the function $\pi_B\big|_G$ is injective.*

iv. *The function f is bijective iff the function $\pi_B\big|_G$ is bijective.*

14. We introduce the notation below:

Let D, R be sets. The **set of all functions from D to R** *is denoted by $\mathsf{Map}(D, R)$.*

(a) Prove the statements below:

i. *Let B be a set. The set $\mathsf{Map}(\emptyset, B)$ is a singleton.*

ii. *Let A, B be sets. Suppose A, B are non-empty. Then $\mathsf{Map}(A, B)$ is non-empty.*

iii. *Let A, B be sets. Suppose A is non-empty. Then $\mathsf{Map}(A, \emptyset) = \emptyset$.*

(b) Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Prove that the statements $(\dagger_1), (\ddagger_1)$ are equivalent:

$(\dagger_1)$ *f is surjective.*

$(\ddagger_1)$ *For any non-empty set C, for any $\varphi, \psi \in \mathsf{Map}(B, C)$, (if $\varphi \circ f = \psi \circ f$ then $\varphi = \psi$).*

(c) Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Prove that the statements $(\dagger_2), (\ddagger_2)$ are equivalent:

$(\dagger_2)$ *f is injective.*

$(\ddagger_2)$ *For any non-empty set C, for any $\varphi, \psi \in \mathsf{Map}(C, A)$, (if $f \circ \varphi = f \circ \psi$ then $\varphi = \psi$).*

(d) Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Prove that the statements $(\dagger_3), (\ddagger_3)$ are equivalent:

$(\dagger_3)$ *f is surjective.*

$(\ddagger_3)$ *For any non-empty set C, for any $\psi \in \mathsf{Map}(C, B)$, there exists some $\varphi \in \mathsf{Map}(C, A)$ such that $f \circ \varphi = \psi$.*

(e) Let A, B be non-empty sets, and $f : A \longrightarrow B$ be a function. Prove that the statements $(\dagger_4), (\ddagger_4)$ are equivalent:

$(\dagger_4)$ *f is injective.*

$(\ddagger_4)$ *For any non-empty set C, for any $\varphi \in \mathsf{Map}(A, C)$, there exists some $\psi \in \mathsf{Map}(B, C)$ such that $\psi \circ f = \varphi$.*