MATH1050 Examples: Formal definition for the notion of functions.

1. (a) Let $F = \{(x, y) \mid y^3 = x^2 + 1 \text{ and } x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$. (Note that $F \subset \mathbb{R}^2$.)
   Verify that $(\mathbb{R}, \mathbb{R}, F)$ is a function.
   (b) Let $A = (0, +\infty)$, and $F = \{(x, y) \mid x^2 y = 1 \text{ and } x \in A \text{ and } y \in A\}$. (Note that $F \subset A^2$.)
   Verify that (A, A, F) is a function.
   (c) Let $A_1 = [0, +\infty)$, $A_2 = (-\infty, 0)$, $F_1 = \{(x, y) \mid y = x^2 \text{ and } x \in A_1 \text{ and } y \in \mathbb{R}\}$, $F_2 = \{(x, y) \mid y = x+1 \text{ and } x \in A_2 \text{ and } y \in \mathbb{R}\}$, and $F = F_1 \cup F_2$. (You can take for granted that $F \subset \mathbb{R}^2$.)
   Verify that $(\mathbb{R}, \mathbb{R}, F)$ is a function.
   Remark. Such is an example of 'piecewise-defined functions' in school mathematics.

2. (a) Let $G = \{(x, y) \mid |y| = |x| \text{ and } x \in \mathbb{R} \text{ and } y \in \mathbb{R}\}$. Is $(\mathbb{R}, \mathbb{R}, G)$ a function? Justify your answer.
   (b) Let $H = \{(x, y) \mid y^2 = x \text{ and } x \in \mathbb{N} \text{ and } y \in \mathbb{N}\}$. Is $(\mathbb{N}, \mathbb{N}, H)$ a function? Justify your answer.
   (c) Let $G = \{(\sqrt{t^2}, \cos(t)) \mid t \in \mathbb{R}\}$. Is $(\mathbb{R}, \mathbb{R}, G)$ a function? Justify your answer.
   (d) Let $H = \{(s^2, s) \mid s \in \mathbb{R}\} \cup \{(t, 0) \mid t \in \mathbb{R} \text{ and } t < 0\}$. Is $(\mathbb{R}, \mathbb{R}, H)$ a function? Justify your answer.

3. In this question, $0, 1, 2$ are regarded as pairwise distinct objects.
   (a) Let $A = \{0, 1, 2\}$, $B = \{0, 1, 2\}$, and $F = \{(0, 1), (1, 2), (2, 1)\}$. Note that $F \subset A \times B$.
   Is (A, B, F) a function? Justify your answer.
   (b) Let $A = \{0, 1\}$, $B = \{0, 1, 2\}$, and $F = \{(0, 1), (0, 2), (1, 0)\}$. Note that $F \subset A \times B$.
   Is (A, B, F) a function? Justify your answer.
   (c) Let $A = \{0, 1, 2\}$, $B = \{0, 1, 2\}$, and $F = \{(0, 0), (1, 2)\}$. Note that $F \subset A \times B$.
   Is (A, B, F) a function? Justify your answer.

4. (a) Let $A = [0, 1]$, $B = [0, 2]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + y^2 = 4\}$. Note that $F \subset A \times B$.
   Define $f = (A, B, F)$. Verify that f is a function.
   (b) Let $A = [0, 1]$, $B = [0, 1]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + y^2 = 4\}$. Note that $F \subset A \times B$.
   Define $f = (A, B, F)$. Is f a function? Justify your answer.
   (c) Let $A = [0, 1]$, $B = [-1, 2]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + y^2 = 4\}$. Note that $F \subset A \times B$.
   Define $f = (A, B, F)$. Is f a function? Justify your answer.

5. (a) Let $A = [0, 3]$, $B = [0, 2]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + 9y^2 = 36\}$. Note that $F \subset A \times B$.
   Define $f = (A, B, F)$. Verify that f is a function.
   (b) Let $A = [0, 3]$, $B = [-1, 2]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + 9y^2 = 36\}$. Note that $F \subset A \times B$.
   Define $f = (A, B, F)$. Is f a function? Justify your answer.
   (c) Let $A = [0, 3]$, $B = [0, 1]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } 4x^2 + 9y^2 = 36\}$. Note that $F \subset A \times B$.
   Define $f = (A, B, F)$. Is f a function? Justify your answer.

6. (a) Let $A = [0, +\infty)$, $F = \{(x, y) \mid x \in A \text{ and } y \in \mathbb{R} \text{ and } (y + 2)^2 = x\}$. Note that $F \subset A \times \mathbb{R}$.
   Is $(A, \mathbb{R}, F)$ a function? Justify your answer.
   (b) Let $A = [0, +\infty)$, $F = \{(x, y) \mid x \in A \text{ and } y \in A \text{ and } (y + 2)^2 = x\}$. Note that $F \subset A^2$.
   Is (A, A, F) a function? Justify your answer.
   (c) Let $A = [0, +\infty)$, $B = [-2, +\infty)$, $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } (y + 2)^2 = x\}$. Note that $F \subset A \times B$.
   Is (A, B, F) a function? Justify your answer.

7. Let $A = [0, 2]$, $B = [0, 3]$, and $F = \{(x, y) \mid x \in A \text{ and } y \in B \text{ and } y^2 = x^2(2-x)\}$. Note that $F \subset A \times B$ by definition.
   Define $f = (A, B, F)$.
   (a) Verify that f is a function.
   (b) Is f injective? Justify your answer.

8. Let $A = [0, +\infty)$. Let $\alpha, \beta \in \mathbb{R}$. Define $C_{\alpha,\beta} = \{(x, y) \mid (y - \beta)^2 = 1 + (x - \alpha)^3\}$.

   (a) Define $E_\alpha = A^2 \cap C_{\alpha,0}$.

      i. For which values of α is (A, A, E_α) a function? Justify your answer.

      ii. For which values of α is (A, A, E_α) an injective function? Justify your answer.

      iii. For which values of α is (A, A, E_α) a surjective function? Justify your answer.

      iv. For which values of α is (A, A, E_α) a bijective function? Justify your answer.

   (b) Write $F_\beta = A^2 \cap C_{0,\beta}$.

      i. For which values of β is (A, A, F_β) a function? Justify your answer.

      ii. For which values of β is (A, A, F_β) an injective function? Justify your answer.

      iii. For which values of β is (A, A, F_β) a surjective function? Justify your answer.

      iv. For which values of β is (A, A, F_β) a bijective function? Justify your answer.

9. Let $A = \{x \in \mathbb{Q} : x = s^3 \text{ for some } s \in \mathbb{Q}\}$, $B = \{y \in \mathbb{Q} : y = t^4 \text{ for some } t \in \mathbb{Q}\}$. Define

$$F = \left\{ (x, y) \;\middle|\; \begin{array}{l} x \in A \text{ and } y \in B \text{ and} \\ \text{there exists some } r \in \mathbb{Q} \text{ such that } (x = r^3 \text{ and } y = r^4). \end{array} \right\},$$

   and $f = (A, B, F)$. Note that $F \subset A \times B$.

   (a) Is f a function from A to B? Justify your answer.

   (b) Where f is a function, write down the 'formula of definition' of f.

   (c) Where f is a function, determine whether f is injective. Justify your answer.

   (d) Where f is a function, determine whether f is surjective. Justify your answer.

10. Consider the 'declarations' below through each of which some function is supposed to be defined. Determine whether it makes sense or not. Justify your answer.

   (a) '*Define the function* $f : [0, 2] \longrightarrow \mathbb{R}$ *by* $\begin{cases} x^2 - x + 2 & \text{if } \; 0 \le x \le 1 \\ x^3 + x - 2 & \text{if } \; 1 \le x \le 2 \end{cases}$.'

   (b) '*Define the function* $f : [0, 4] \longrightarrow \mathbb{R}$ *by* $\begin{cases} \ln(x + 1) & \text{if } \; 0 \le x < 2 \\ \log_{x-1}(x) & \text{if } \; 2 < x \le 4 \end{cases}$.'

11. Consider the 'declarations' below through each of which some function is supposed to be defined. Determine whether it makes sense or not. Justify your answer.

   (a) '*Define the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(t^4) = t^3$ *for any* $t \in \mathbb{R}$.'

   (b) '*Define the function* $f : (0, +\infty) \longrightarrow \mathbb{R}$ *by* $f(t/s) = \dfrac{t^2 + 1}{s^2 + 1}$ *for any* $s, t \in (0, +\infty)$.'

12. For each $n \in \mathbb{N}\backslash\{0\}$, we define $\omega_n = \cos\left(\dfrac{2\pi}{n}\right) + i \sin\left(\dfrac{2\pi}{n}\right)$, and $Z_n = \{\zeta \in \mathbb{C} : \zeta = {\omega_n}^k \text{ for some } k \in \mathbb{Z}\}$.

   You may take for granted that ${\omega_n}^n = 1$ *and* $Z_n = \{1, \omega_n, {\omega_n}^2, \cdots, {\omega_n}^{n-1}\}$. (Note that Z_n is the set of all n-th roots of unity.)

   Consider the 'declarations' below through each of which some function is supposed to be defined. Determine whether it makes sense or not. Justify your answer.

   (a) '*Define the function* $f : Z_6 \longrightarrow \mathbb{Z}$ *by* $f({\omega_6}^k) = 6k$ *for any* $k \in \mathbb{Z}$.'

   (b) '*Define the function* $f : Z_{12} \longrightarrow Z_6$ *by* $f({\omega_{12}}^k) = {\omega_6}^k$ *for any* $k \in \mathbb{Z}$.'

13. For any $k \in \mathbb{N}\backslash\{0\}$, define $\omega_k = \cos\left(\dfrac{2\pi}{k}\right) + i \sin\left(\dfrac{2\pi}{k}\right)$, and define $Z_k = \{\zeta \in \mathbb{C} : \zeta = {\omega_k}^j \text{ for some } j \in \mathbb{Z}\}$.

   Let $m, n \in \mathbb{N}\backslash\{0\}$. Define the subset $F_{m,n}$ of $Z_m \times Z_n$ by

$$F_{m,n} = \left\{ (\zeta, \eta) \;\middle|\; \text{There exist some } r \in \mathbb{Z} \text{ such that } \zeta = {\omega_m}^r \text{ and } \eta = {\omega_n}^r \right\}.$$

   Define the relation $f_{m,n}$ by $f_{m,n} = (Z_m, Z_n, F_{m,n})$. Consider the statements $(\star), (\star\star)$ below:

($\star$) *m is divisible by n.*

($\star\star$) *$f_{m,n}$ is a function from Z_m to Z_n.*

(a) Suppose ($\star$) holds. Prove that ($\star\star$) holds.

(b) Suppose ($\star\star$) holds. Prove that ($\star$) holds.

14. Let $A = \mathbb{C}\backslash\{1\}$, $F = \{(x, y) \mid x \in A \text{ and } y \in A \text{ and } (x-1)y = x\}$. Define $f = (A, A, F)$. Note that $F \subset A^2$.

    (a) Verify that f is a function.

    (b) What is the 'formula of definition' of the function f.

    (c) Verify that the function f is bijective.

    (d) What is the 'formula of definition' of the inverse function of f.

    (e) What are the respective 'formulae of definition' of $f \circ f$ and $f \circ f \circ f$?

15. Let $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. Define

$$F = \left\{ \ (z, w) \mid z \in \mathbb{C} \text{ and } w \in D \text{ and } w = \frac{z}{\sqrt{1+|z|^2}}(\cos(|z|) + i\sin(|z|)) \ \right\}.$$

Note that $F \subset \mathbb{C} \times D$. Define $f = (\mathbb{C}, D, F)$.

(a) Verify that f is a function.

(b) Is f injective? Justify your answer.

(c) Let $w \in D$, and $\theta \in \mathbb{R}$. Define the complex number v by $v = \dfrac{w}{\sqrt{1-|w|^2}}(\cos(\theta) + i\sin(\theta))$.

Verify that

$$f(v) = Kw(\cos(L\theta + M \cdot \frac{|w|}{\sqrt{1-|w|^2}}) + i\sin(L\theta + M \cdot \frac{|w|}{\sqrt{1-|w|^2}}))$$

Here K, L, M are some appropriate non-negative integers whose values are independent of that of w and θ. You have to determine the value of K, L, M explicitly.

(d) Is f bijective? Justify your answer.