MATH1050 Examples: Image sets and pre-image sets.

1. Let $A = \{0, 1, 2, 3, 4\}$, $B = \{5, 6, 7, 8, 9\}$. Define the function $f : A \longrightarrow B$ by $f(0) = 5$, $f(1) = 5$, $f(2) = 7$, $f(3) = 9$, $f(4) = 9$.

   Consider each of the sets below. Where it is not the empty set, list every element of the set concerned, each element exactly once. Where it is the empty set, write '*it is the empty set*'.

   (a) $f(\emptyset)$
   (b) $f(\{0\})$
   (c) $f(\{0, 1\})$
   (d) $f(\{0, 1, 2\})$
   (e) $f(\{0, 2, 4\})$
   (f) $f(A)$
   (g) $f^{-1}(\emptyset)$
   (h) $f^{-1}(\{5\})$
   (i) $f^{-1}(\{6\})$
   (j) $f^{-1}(\{5, 6\})$
   (k) $f^{-1}(\{5, 6, 7\})$
   (l) $f^{-1}(B)$

2. Define the function $f : [-1, 2] \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} 1 & \text{if} \quad x = -1 \\ x + 1 & \text{if} \quad -1 < x < 0 \\ 2x & \text{if} \quad 0 \leq x \leq 1 \\ -x + 2 & \text{if} \quad 1 < x \leq 2 \end{cases}$$

   Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \mu, \nu, \sigma, \tau$, so that the set equalities in parts (a), (b), (c), (d) hold. You are not required to justify your answer. (But it may help if you draw the graph of f first.)

   (a) $f^{-1}(\{0\}) = \{\alpha, 2\}$.
   (b) $f([1, 2]) = [0, \beta) \cup \{\gamma\}$.
   (c) $f^{-1}([0.5, 3]) = \{\delta\} \cup [-0.5, \varepsilon) \cup [\zeta, 1.5]$.
   (d) $f^{-1}(f([0, 0.5])) = [\mu, \nu] \cup (\sigma, \tau]$.

3. Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by

$$f(x) = \begin{cases} x + 3 & \text{if} \quad x < -1 \\ -3 & \text{if} \quad x = -1 \\ -2x - 2 & \text{if} \quad -1 < x < 0 \\ 3 & \text{if} \quad x = 0 \\ \dfrac{5}{5x + 1} & \text{if} \quad x > 0 \end{cases}.$$

   Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta$, so that the set equalities below hold. You are not required to justify your answer. (It may help if you draw the graph of f first.)

   (a) $f((-3, 0)) = ((\alpha, \beta) \backslash \{\gamma\}) \cup \{\delta\}$.
   (b) $f^{-1}([-1, 4]) = ([-4, \varepsilon] \backslash \{\zeta\}) \cup \{\eta\} \cup [\theta, +\infty)$.

4. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $g(x) = \begin{cases} x & \text{if} \quad x \in \mathbb{Q} \\ 0 & \text{if} \quad x \in \mathbb{R} \backslash \mathbb{Q} \end{cases}$.

   Write down the respective values of the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta, \eta, \theta, \kappa, \lambda, \mu, \nu, \xi, \rho, \sigma, \tau, \varphi, \psi, \omega$, so that the set equalities below hold. You are not required to justify your answer.

   (a) $g(\{1\}) = \{\alpha\}$.
   (b) $g(\{\sqrt{2}\}) = \{\beta\}$.
   (c) $g(\{1, \sqrt{2}, \sqrt{3}\}) = \{\gamma, \delta\}$.
   (d) $g^{-1}(\{1\}) = \{\varepsilon\}$.
   (e) $g^{-1}(\{1, \sqrt{2}\}) = \{\zeta\}$.
   (f) $g^{-1}(\{0, 1\}) = \{\eta, \theta\} \cup (\mathbb{R} \backslash \mathbb{Q})$.
   (g) $g([1, 2]) = \{\kappa\} \cup ([\lambda, \mu] \cap \mathbb{Q})$.
   (h) $g^{-1}(g([1, 2])) = \{\nu\} \cup [\xi, \rho] \cup (\mathbb{R} \backslash \mathbb{Q})$.
   (i) $g^{-1}([1, 3]) = [\sigma, \tau] \cap \mathbb{Q}$.
   (j) $g(g^{-1}([1, 3])) = [\varphi, \psi] \cap \mathbb{Q}$.

5. *You are not required to justify your answers in this question.*

Let $c \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} |(x+5)(x-3)| + 2 & \text{if} \quad x \leq 0 \\ \dfrac{c}{1+x^2} & \text{if} \quad x > 0 \end{cases}$$

Suppose $f^{-1}(\{v\}) \neq \emptyset$ for any $v \in (0, +\infty)$. Further suppose $f^{-1}(\{2\})$ has exactly one element.

(a) What is the value of c?

(b) Name the only element of $f^{-1}(\{2\})$.

(c) What are the numbers $\alpha, \beta, \gamma, \delta$ for which the set equality $f([-3, 3]) = [\alpha, \beta) \cup [\gamma, \delta]$ holds?

6. *You are not required to justify your answers in this question.*

Let $a, b \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by

$$f(x) = \begin{cases} \dfrac{2}{1+x^2} + b & \text{if} \quad x \leq 0 \\ x & \text{if} \quad 0 < x < 2 \\ a & \text{if} \quad x = 2 \\ \dfrac{1}{x-2} & \text{if} \quad x > 2 \end{cases}$$

Suppose $f(\mathbb{R}) = [0, +\infty)$. Further suppose that the set $f^{-1}(\{1\})$ has exact two elements, and for any $v \in (3, +\infty)$, the set $f^{-1}(\{v\})$ is a singleton.

(a) What is the value of a?

(b) Write down both elements of $f^{-1}(\{1\})$.

(c) What is the value of b?

(d) What are the numbers $\alpha, \beta, \gamma, \delta, \varepsilon, \zeta$ for which the set equality $f^{-1}((1.5, 2]) = (\alpha, \beta] \cup (\gamma, \delta) \cup [\varepsilon, \zeta)$ holds?

7. Let $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{x^2}$ for any $x \in \mathbb{R}\backslash\{0\}$.

(a) Consider the sets below. Express each of them as an interval or a union of several 'disjoint' intervals.

i. $f([1, 2])$

ii. $f((0, 1))$

iii. $f^{-1}([1, 4])$

iv. $f^{-1}([-1, 1])$

(b) Prove the set equalities you have written down.

8. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{8x}{x^2+4}$ for any $x \in \mathbb{R}$. Note that f is differentiable on $\mathbb{R}$.

(a) i. Find the respective value of $\lim\limits_{t \to -\infty} f(t)$, $\lim\limits_{t \to +\infty} f(t)$.

ii. Determine where f is strictly increasing, and where f is strictly decreasing.

iii. Hence, or otherwise, determine where f attains any relative maximum or any relative minimum, if such exist.

iv. Hence, or otherwise, determine where f attains the absolute maximum or the absolute minimum, if such exists.

Remark. With these information, you should be able to sketch the graph of f.

(b) i. Consider the sets below. Express each of them as an interval or a union of several 'disjoint' intervals.

A. $f(\mathbb{R})$

B. $f([0, 1])$

C. $f([1, 3])$

D. $f([1, +\infty))$

ii. Prove the set equalities you have written down.

Remark. The difficulty is in the juggling of inequalities (involving fractions and surd forms) together with quantifiers.

(c) i. Consider the sets below. Express each of them as an interval or a union of several 'disjoint' intervals.

A. $f^{-1}(\mathbb{R})$

B. $f^{-1}([0, 2])$

C. $f^{-1}((1, 3))$

D. $f^{-1}((-1, 1))$

ii. Prove the set equalities you have written down.

9. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}^2$ be defined by $f(t) = \left(\dfrac{2t}{t^2+1}, \dfrac{t^2-1}{t^2+1}\right)$ for any $t \in \mathbb{R}$.

Let $C = \{(u, v) \mid u, v \in \mathbb{R} \text{ and } u^2 + v^2 = 1\}$ and $p = (0, 1)$.

i. Verify that f is injective.

ii. Verify that $f(\mathbb{R}) = C\backslash\{p\}$.

Remark. Hence the function f parametrizes the circle C with the point p removed. It is the 'baby version' of what is known as the **stereographic projection** for the (n-dimensional) sphere. The stereographic projection for the (2-dimensional) sphere is described in the next part, with $\mathbb{R}^2$ being identified as $\mathbb{C}$.

(b) Let $g : \mathbb{C} \longrightarrow \mathbb{R}^3$ be defined by $g(z) = \left(\dfrac{2\mathsf{Re}(z)}{|z|^2+1}, \dfrac{2\mathsf{Im}(z)}{|z|^2+1}, \dfrac{|z|^2-1}{|z|^2+1}\right)$ for any $z \in \mathbb{C}$.

Let $S = \{(u, v, w) \mid u, v, w \in \mathbb{R} \text{ and } u^2 + v^2 + w^2 = 1\}$ and $p = (0, 0, 1)$.

i. Verify that g is injective.

ii. Verify that $g(\mathbb{C}) = S\backslash\{p\}$.

10. Let $r > 0$ and $R > 0$. Suppose $R > r$.

Let $f : (-1, 1) \times (-1, 1) \longrightarrow \mathbb{R}^3$ be defined by $f(s, t) = ((R + r\cos(\pi t))\cos(\pi s), (R + r\cos(\pi t))\sin(\pi s), r\sin(\pi t))$ for any $s, t \in (-1, 1)$.

Let $T = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } (\sqrt{x^2+y^2} - R)^2 + z^2 = r^2\}$, $C_1 = \{(x, 0, z) \mid x, z \in \mathbb{R} \text{ and } (x + R)^2 + z^2 = r^2\}$, $C_2 = \{(x, y, 0) \mid x, y \in \mathbb{R} \text{ and } x^2 + y^2 = (R - r)^2\}$.

(a) Verify that f is injective.

(b) Verify that $f((-1, 1) \times (-1, 1)) = T\backslash(C_1 \cup C_2)$.

Remark. T is a 'torus' in $\mathbb{R}^3$ with centre at the origin, with the z-axis as the axis of symmetry, and the xy-plane as the plane of symmetry. It is the surface of revolution obtained by rotating the circle $(x - R)^2 + z^2 = r^2$ on the xz-plane around the z-axis.

11. Let $f, g, h : \mathbb{R}^2 \longrightarrow \mathbb{R}$ be functions defined by $f(x, y) = |x| + |y|$, $g(x, y) = \sqrt{x^2 + y^2}$, $h(x, y) = \max(|x|, |y|)$ for any $x, y \in \mathbb{R}$.

Let $I = [0, 1]$. Prove that $f^{-1}(I) \subsetneq g^{-1}(I) \subsetneq h^{-1}(I)$.

Remark. It helps to sketch the 'contour map' on $\mathbb{R}^2$ for the functions f, g, h first.

12. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2$ for any $z \in \mathbb{C}$. Let $L = \{z \in \mathbb{C} : \mathsf{Re}(z) = 1\}$.

(a) Let $\gamma : \mathbb{R} \longrightarrow \mathbb{C}$ be the function defined by $\gamma(t) = 1 + ti$ for any $t \in \mathbb{R}$.

i. Prove that $\gamma(\mathbb{R}) = L$.

ii. Find the 'explicit formula of definition' for the function $f \circ \gamma$.

iii. Hence, or otherwise, prove that $f(L)$ is the set of all points of a certain parabola on the Argand plane. Also give the equation for that parabola.

(b) i. Let $t \in \mathbb{R}$. Suppose $z = \zeta$ is a solution of the equation $z^2 = 1 + ti$ with unknonwn $z \in \mathbb{C}$. Prove that $(\mathsf{Re}(\zeta))^2 - (\mathsf{Im}(\zeta))^2 = 1$ and $2\mathsf{Re}(\zeta)\mathsf{Im}(\zeta) = t$.

ii. Prove that $f^{-1}(L)$ is the set of all points of a certain hyperbola (with two branches) on the Argand plane. Also give the equation for that hyperbola.

Remark. How about replacing L by some other straight line on the Argand plane? (Be careful with the difference between the case in which the straight line passes through 0 and the case in which the straight line does not pass through 0.)

13. Let $a \in \mathbb{R}\backslash\{0\}$, $A = \{w \in \mathbb{C} : \mathsf{Im}(w) = a\}$, and $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ be the function defined by $f(z) = \dfrac{1}{z}$ for any $z \in \mathbb{C}\backslash\{0\}$.

    (a) Prove that $f^{-1}(A) = \left\{z \in \mathbb{C}\backslash\{0\} : (\mathsf{Re}(z))^2 + (\mathsf{Im}(z))^2 + \dfrac{\mathsf{Im}(z)}{a} = 0\right\}$.

    (b) Sketch the picture of $f^{-1}(A)$ as a 'geometric figure' on the 'Argand plane' of $\mathbb{C}$.

14. Denote by $C^\infty(\mathbb{R})$ the set of all smooth real-valued functions of one real variable whose respective domains are $\mathbb{R}$.

    For each $n \in \mathbb{N}$, denote by P_n the set of all degree-n polynomial functions from $\mathbb{R}$ to $\mathbb{R}$ with real coefficients.

    Define the function $D : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ by $D(\varphi) = \varphi'$ for any $\varphi \in C^\infty(\mathbb{R})$.

    Verify that $D(P_{n+1}) = P_n$ for any $n \in \mathbb{N}$.

15. *Familiarity with linear algebra is assumed in this question.*

    We introduce the following definition:

    - *Let S be a subset of $\mathbb{R}^n$. The set S is said to be* **convex** *if for any $p, q \in \mathbb{R}^n$, for any $t \in [0, 1]$, $tp + (1-t)q \in S$.*

    Let H be an $m \times n$-matrix with real entries, and $L_H : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be the function defined by $L_H(x) = Hx$ for any $x \in \mathbb{R}^n$. (Note that L_H is a linear transformation: we have $L_H(ax + by) = aL_H(x) + bL_H(y)$ for any $x, y \in \mathbb{R}^n$.)

    (a) Prove that $L_H(S)$ is a convex subset of $\mathbb{R}^m$ for any convex subset S of $\mathbb{R}^n$.

    (b) Prove that ${L_H}^{-1}(U)$ is a convex subset of $\mathbb{R}^n$ for any convex subset U of $\mathbb{R}^m$.