MATH1050 Examples: Surjectivity, injectivity, and inverse functions.

1. Let $f : [0,9] \longrightarrow \mathbb{R}$ be the function defined by $f(x) = -x + 6\sqrt{x} - 5$ for any $x \in [0,9]$.
   (a) Show that $f(x) = -(A - \sqrt{x})^2 + B$ for any $x \in [0,9]$. Here A, B are some real constants whose respective values you have to determine.
   (b) Verify that f is injective directly from the definition of injectivity.
   (c) Is f surjective? Justify your answer directly from the definition of surjectivity.

2. Let $f : (0,+\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{x^2-1}{x^2+1}\sin\left(\dfrac{1}{\sqrt{x}}\right)$ for any $x \in (0,+\infty)$.
   (a) Verify that f is not injective.
   (b) i. Verify that $\left|\dfrac{x^2-1}{x^2+1}\right| \leq 1$ for any $x \in (0,+\infty)$.
   Remark. A very simple answer can be obtained without using calculus.
   ii. Apply the previous part, or otherwise, to verify that f is not surjective.

3. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 2x+5 & \text{if } x \leq -3 \\ -x & \text{if } -3 < x < 1 \\ x-3 & \text{if } x \geq 1 \end{cases}$.
   (a) Is f surjective? Why?
   (b) Is f injective? Why?

4. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} x & \text{if } x \geq 0 \\ -2x & \text{if } x < 0 \end{cases}$.
   (a) Is f surjective? Why?
   (b) Is f injective? Why?

5. Let $f : (0,+\infty) \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} x+2 & \text{if } x > 0 \text{ and } x \text{ is rational} \\ 2x-1 & \text{if } x > 0 \text{ and } x \text{ is irrational} \end{cases}$.
   (a) Is f surjective? Why?
   (b) Is f injective? Why?

6. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \begin{cases} 0 & \text{if } x \in \mathbb{Q} \\ x^2 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.
   (a) Is f injective? Justify your answer.
   (b) Is f surjective? Justify your answer.

7. Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions respectively defined by $f(x) = \begin{cases} x & \text{if } x \in \mathbb{Q} \\ x^3 & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$, $g(x) = \begin{cases} x^3 & \text{if } x \in \mathbb{Q} \\ x & \text{if } x \in \mathbb{R}\backslash\mathbb{Q} \end{cases}$.
   (a) i. Is f injective? Justify your answer.
   ii. Is f surjective? Justify your answer.
   (b) i. Is g injective? Justify your answer.
   ii. Is g surjective? Justify your answer.

8. (a) Let $A = (0,+\infty)$, and $f : A \longrightarrow A$ be the function defined by $f(x) = \dfrac{1}{x^2}$ for any $x \in A$.
   i. Verify that f is surjective.
   ii. Verify that f is injective.
   (b) Let $B = (0,+\infty) \cap \mathbb{Q}$, and $g : B \longrightarrow B$ be the function defined by $g(x) = \dfrac{1}{x^2}$ for any $x \in B$.

i. Verify that g is not surjective.
ii. Is g injective? Why?

9. (a) Let $f : \mathbb{R}\backslash\{-1,1\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = \dfrac{1}{x^2-1}$ for any $x \in \mathbb{R}\backslash\{-1,1\}$.

   i. Verify that f is not injective.
   ii. Verify that f is not surjective.

   (b) i. Let $x \in \mathbb{R}\backslash\{-1,1\}$. Verify that $\dfrac{1}{1-x^2} \in \mathbb{R}\backslash(-1,0]$.

   ii. Let $y \in \mathbb{R}\backslash(-1,0]$. Verify that there exists some $x \in \mathbb{R}\backslash\{-1,1\}$ such that $y = \dfrac{1}{1-x^2}$.

   iii. Let $g : \mathbb{R}\backslash\{-1,1\} \longrightarrow \mathbb{R}\backslash(-1,0]$ be the function defined by $g(x) = \dfrac{1}{x^2-1}$ for any $x \in \mathbb{R}\backslash\{-1,1\}$.
   Is g surjective? Why?

   Remark. What is the point of this question? Starting from a non-surjective function, if we 'restrict' its range appropriately, we will obtain a new function which is surjective. However, we need be careful not to 'over-restrict' it.

10. (a) Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^2 - 2x$ for any $x \in \mathbb{R}$.
    i. Is f injective? Justify your answer.
    ii. Is f surjective? Justify your answer.

    (b) Verify that for any $x \in (1, +\infty)$, $x^2 - 2x > -1$.

    (c) Let $g : (1, +\infty) \longrightarrow (-1, +\infty)$ be the function defined by $g(x) = x^2 - 2x$ for any $x \in (1, +\infty)$.
    i. Is g injective? Justify your answer.
    ii. Is g surjective? Justify your answer.
    iii. Is g bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

11. Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $g(x) = \dfrac{10^x - 10^{-x}}{2}$ for any $x \in \mathbb{R}$.

    (a) Verify that g is injective.
    (b) Verify that g is surjective.
    (c) What is the 'formula of definition' of the function $g^{-1} : \mathbb{R} \longrightarrow \mathbb{R}$?

    Remark. There is no need to use any results from the calculus in this question.

12. (a) Prove that for any $t \in \mathbb{R}$, $0 < \dfrac{1}{\sqrt{1+e^{-t}}} < 1$.

    (b) Denote the interval $(0, 1)$ by I. Define the function $g : \mathbb{R} \longrightarrow I$ by $g(x) = \dfrac{1}{\sqrt{1+e^{-x}}}$ for any $x \in \mathbb{R}$.

    i. Verify that g is surjective, directly from the definition of surjectivity.
    ii. Verify that g is injective, directly from the definition of injectivity.
    iii. Is g bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

13. (a) Prove that for any $x \in \mathbb{R}\backslash\{2\}$, $\dfrac{3x}{x-2} \neq 3$.

    (b) Let $f : \mathbb{R}\backslash\{2\} \longrightarrow \mathbb{R}\backslash\{3\}$ be the function defined by $f(x) = \dfrac{3x}{x-2}$ for any $x \in \mathbb{R}\backslash\{2\}$.

    i. Is f injective? Justify your answer.
    ii. Is f surjective? Justify your answer.
    iii. Is f bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

14. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x|x|$ for any $x \in \mathbb{R}$.

    (a) Is f injective? Justify your answer.
    (b) Is f surjective? Justify your answer.
    (c) Is f bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

15. Let $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x\sqrt{|x|}$ for any $x \in \mathbb{R}$.

    (a) Is f injective? Justify your answer.

    (b) Is f surjective? Justify your answer.

    (c) Is f bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

16. (a) Prove that for any $x \in \mathbb{R}$, $-1 < \dfrac{x|x|}{x^2+1} < 1$.

    (b) Let $f : \mathbb{R} \longrightarrow (-1, 1)$ be the function defined by $f(x) = \dfrac{x|x|}{x^2+1}$ for any $x \in \mathbb{R}$.

    i. Is f injective? Justify your answer.
    ii. Is f surjective? Justify your answer.
    iii. Is f bijective? If *yes*, also write down the 'formula of definition' for its inverse function.

17. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2 - 6z + 13$ for any $z \in \mathbb{C}$.

    (a) Verify that f is surjective.

    (b) Verify that f is not injective.

18. Let $f : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}\backslash\{0\}$ be the function defined by $f(z) = \dfrac{z^2}{\bar{z}}$ for any $z \in \mathbb{C}\backslash\{0\}$.

    (a) Verify that $f(z) = \dfrac{z^3}{|z|^2}$ for any $z \in \mathbb{C}\backslash\{0\}$.

    (b) Is f injective? Justify your answer.

    (c) Is f surjective? Justify your answer.

19. (a) Let $f : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x + \dfrac{1}{x}$ for any $x \in \mathbb{R}\backslash\{0\}$.

    i. Verify that f is not surjective.
    ii. Verify that f is not injective.

    (b) Verify that for any $x \in \mathbb{R}\backslash\{0\}$, $\left|x + \dfrac{1}{x}\right| > 2$.

    (c) Let $g : \mathbb{R}\backslash\{0\} \longrightarrow \mathbb{R}\backslash(-2, 2)$ be the function defined by $g(x) = x + \dfrac{1}{x}$ for any $x \in \mathbb{R}\backslash\{0\}$.

    i. Verify that g is surjective.
    ii. Is g injective? Why?

    (d) Let $h : \mathbb{C}\backslash\{0\} \longrightarrow \mathbb{C}$ be the function defined by $h(z) = z + \dfrac{1}{z}$ for any $z \in \mathbb{C}\backslash\{0\}$.

    i. Verify that h is surjective.
    ii. Is h injective? Why?

20. Let $A = \{z \in \mathbb{C} : -\pi < \mathsf{Im}(z) < \pi\}$, $L = \{w \in \mathbb{C} : \mathsf{Im}(w) = 0 \text{ and } \mathsf{Re}(w) \leq 0\}$, and $B = \mathbb{C}\backslash L$.

    (a) Prove that for any $z \in A$, $e^{\mathsf{Re}(z)}(\cos(\mathsf{Im}(z)) + i\sin(\mathsf{Im}(z))) \in B$.

    (b) Define the function $f : A \longrightarrow B$ by $f(z) = e^{\mathsf{Re}(z)}(\cos(\mathsf{Im}(z)) + i\sin(\mathsf{Im}(z)))$ for any $z \in A$.

    i. Verify that f is injective.
    ii. Verify that f is surjective.

21. (a) Prove that for any $z \in \mathbb{C}\backslash\{-2\}$, $\dfrac{z+i}{z+2} \neq 1$.

    (b) Let $f : \mathbb{C}\backslash\{-2\} \longrightarrow \mathbb{C}\backslash\{1\}$ be the function defined by $f(z) = \dfrac{z+i}{z+2}$ for any $z \in \mathbb{C}\backslash\{-2\}$.

    i. Prove that f is injective.
    ii. Prove that f is surjecive.
    iii. What is the 'formula of definition' of the inverse function of f, namely, the function $f^{-1} : \mathbb{C}\backslash\{1\} \longrightarrow \mathbb{C}\backslash\{-2\}$?

22. For any $a, b \in \mathbb{C}$, define the function $f_{a,b} : \mathbb{C} \longrightarrow \mathbb{C}$ by $f_{a,b}(z) = az + b\bar{z}$ for any $z \in \mathbb{C}$.

   (a) Let $a, b, c, d \in \mathbb{C}$.

      i. Verify that $f_{c,d}(f_{a,b}(z)) = (ac + \bar{b}d)z + (bd + \bar{a}d)\bar{z}$ for any $z \in \mathbb{C}$.

      ii. Verify that $f_{ac,bc}(z) = cf_{a,b}(z)$ for any $z \in \mathbb{C}$.

      iii. Suppose $c \in \mathbb{R}$. Verify that $f_{ac,bc}(z) = f_{a,b}(cz)$ for any $z \in \mathbb{C}$.

   (b) Let $a, b \in \mathbb{C}$. Prove that there exist some $\alpha, \beta \in \mathbb{C}$ such that $f_{\alpha,\beta}(f_{a,b}(z)) = (|a|^2 - |b|^2)z$ for any $z \in \mathbb{C}$.

   Remark. Make use of part (a.i) to find candidates for α, β.

   (c) Let $a, b \in \mathbb{C}$. Suppose $|a| \neq |b|$. Prove that $f_{a,b}$ is bijective. What is the 'formula of definition' of the inverse function of $f_{a,b}$?

   Remark. Instead of checking surjectivity and injectivity directly from definition, make use of parts (b), (a.ii), (a.iii) to write down a candidate inverse function for the function $f_{a,b}$.

   (d) Let $a, b \in \mathbb{C}$. Suppose $|a| = |b|$. Is $f_{a,b}$ bijective? Justify your answer.

23. Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z(\mathsf{Re}(z))^2 + i\mathsf{Im}(z)$ for any $z \in \mathbb{C}$.

   (a) Verify that $\begin{cases} \mathsf{Re}(f(z)) &= A(\mathsf{Re}(z))^M \\ \mathsf{Im}(f(z)) &= [B(\mathsf{Re}(z))^N + C] \cdot \mathsf{Im}(z) \end{cases}$ for any $z \in \mathbb{C}$.

   Here A, B, C, M, N are integers whose values you have to determine explicitly.

   (b) Verify that f is injective, directly from the definition of injectivity.

   (c) Verify that f is surjective, directly from the definition of surjectivity.

   (d) Write down the 'formula of definition' of the inverse function $f^{-1} : \mathbb{C} \longrightarrow \mathbb{C}$ of the function f.

24. (a) Let $n \in \mathbb{N}\backslash\{0\}$, and $a \in \mathbb{C}\backslash\{0\}$. Define the function $\mu : \mathbb{C} \longrightarrow \mathbb{C}$ by $\mu(z) = az^n$ for any $z \in \mathbb{C}$. Prove that μ is bijective iff $n = 1$.

   (b) Let $h : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by

$$h(z) = \begin{cases} iz & \text{if} \quad |z| \in \mathbb{Q} \\ \dfrac{3i}{2\bar{z}} & \text{if} \quad |z| \in \mathbb{R}\backslash\mathbb{Q} \end{cases}.$$

      i. Prove the statement below:
         - *For any $\zeta \in \mathbb{C}$, if $|\zeta|$ is irrational then $|h(\zeta)|$ is irrational.*

      ii. Prove that $h \circ h$ is a polynomial function on $\mathbb{C}$. Determine the explicit formula of definition for $h \circ h$ as well.

      iii. Is h bijective? Justify your answer. (*Hint.* Make good use of the result in the previous part.)

25. We introduce this definition:

   A **monic polynomial** *is a polynomial whose leading coefficient is* 1.

   Denote by M the set of all monic quadratic polynomials with real coefficients and indeterminate x.

   (a) For any $\alpha, \beta \in \mathbb{R}$, denote by $p_{\alpha,\beta}(x)$ the monic quadratic polynomial $(x - \alpha)(x - \beta)$ with indeterminate x. Define the function $\Phi : \mathbb{R}^2 \longrightarrow M$ by $(\alpha, \beta) \underset{\Phi}{\longmapsto} p_{\alpha,\beta}(x)$ for any $\alpha, \beta \in \mathbb{R}$.

      i. Is Φ surjective? Justify your answer.

      ii. Is Φ injective? Justify your answer.

   (b) For any $\alpha, \beta \in \mathbb{R}$, denote by $q_{\alpha,\beta}(x)$ the monic quadratic polynomial $(x - \alpha)^2 + \beta$ with indeterminate x. Define the function $\Psi : \mathbb{R}^2 \longrightarrow M$ by $(\alpha, \beta) \underset{\Psi}{\longmapsto} q_{\alpha,\beta}(x)$ for any $\alpha, \beta \in \mathbb{R}$.

      i. Is Ψ surjective? Justify your answer.

      ii. Is Ψ injective? Justify your answer.

26. Consider each of the statements below. For each of them, dis-prove it by constructing an appropriate counter-example.

   (a) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose f, g are injective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$ is injective.*

(b) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose f, g are injective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$ is not injective.*

(c) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose f, g are injective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x)g(x)$ for any $x \in \mathbb{R}$ is injective.*

(d) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose f, g are injective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x)g(x)$ for any $x \in \mathbb{R}$ is not injective.*

(e) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose f, g are surjective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$ is surjective.*

(f) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose f, g are surjective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x) + g(x)$ for any $x \in \mathbb{R}$ is not surjective.*

(g) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose f, g are surjective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x)g(x)$ for any $x \in \mathbb{R}$ is surjective.*

(h) *Let $f, g : \mathbb{R} \longrightarrow \mathbb{R}$ be functions. Suppose f, g are surjective. Then the function $h : \mathbb{R} \longrightarrow \mathbb{R}$ defined by $h(x) = f(x)g(x)$ for any $x \in \mathbb{R}$ is not surjective.*

27. Let J be an open interval in $\mathbb{R}$. (It is assumed that J is not the empty set.) Denote by $C(J)$ the set of all real-valued continuous functions on J. Denote by $C^1(J)$ the set of all real-valued differentiable functions on J whose first derivatives are continuous functions on J.

Define the function $D : C^1(J) \longrightarrow C(J)$ by $D(\varphi) = \varphi'$ for any $\varphi \in C^1(J)$.

For each $a \in J$, define the function $I_a : C(J) \longrightarrow C^1(J)$ by $(I_a(\psi))(x) = \int_a^x \psi(t)dt$ for any $\psi \in C(J)$ for any $x \in J$.

(a) i. Is the function $D : C^1(J) \longrightarrow C(J)$ surjective?

ii. Is the function $D : C^1(J) \longrightarrow C(J)$ injective?

(b) Let $a \in J$.

i. Prove that $((I_a \circ D)(\varphi))(x) = \varphi(x) - \varphi(a)$ for any $\varphi \in C^1(J)$ for any $x \in J$.

ii. Prove that $((D \circ I_a)(\psi))(x) = \psi(x)$ for any $\psi \in C(J)$ for any $x \in J$.

(c) Let $a \in J$.

i. Is the function $I_a : C(J) \longrightarrow C^1(J)$ surjective?

ii. Is the function $I_a : C(J) \longrightarrow C^1(J)$ injective?

iii. Is the function $I_a \circ D : C^1(J) \longrightarrow C^1(J)$ surjective?

iv. Is the function $I_a \circ D : C^1(J) \longrightarrow C^1(J)$ injective?

28. We recall this definition from the calculus of one real variable:

- *A real-valued function of one real variable is said to be* **smooth** *if it is differentiable for as many times as we like at every point of its domain.*

Denote by $C^\infty(\mathbb{R})$ the set of all smooth functions on $\mathbb{R}$.

Let $X : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be the function defined by $(X(\varphi))(x) = x\varphi(x)$ for any $\varphi \in C^\infty(\mathbb{R})$ for any $x \in \mathbb{R}$.

Let $D : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be the function defined by $(D(\varphi))(x) = \varphi'(x)$ for any $\varphi \in C^\infty(\mathbb{R})$ for any $x \in \mathbb{R}$.

Let $I_0 : C^\infty(\mathbb{R}) \longrightarrow C^\infty(\mathbb{R})$ be the function defined by $(I_0(\varphi))(x) = \int_0^x \varphi(t)dt$ for any $\varphi \in C^\infty(\mathbb{R})$ for any $x \in \mathbb{R}$.

(a) Verify that for any $\varphi \in C^\infty(\mathbb{R})$, for any $x \in \mathbb{R}$, $((D \circ X)(\varphi))(x) - (X \circ D)(\varphi))(x) = \varphi(x)$.

Remark. This seemingly innocuous mathematical statement is a baby case of something of great significance in *modern physics*; it is behind the **Heisenberg relations** for position and momentum in *quantum mechanics*.

(b) Verify that for any $\varphi \in C^\infty(\mathbb{R})$, for any $x \in \mathbb{R}$, $((X \circ I_0)(\varphi))(x) - (I_0 \circ X)(\varphi))(x) = ((I_0 \circ I_0)(\varphi))(x)$.

29. *In this question, you are supposed to be familiar with the notion of continuity in the calculus of one real variable. You may take for granted the validity of* **Bolzano's Intermediate Value Theorem**.

- *Let $a, b \in \mathbb{R}$, with $a < b$, and $f : [a, b] \longrightarrow \mathbb{R}$ be a function. Suppose f is continuous on $[a, b]$. Further suppose $f(a)f(b) < 0$. Then f has a zero in (a, b).*

(a) Let $p, q, r, s, t \in \mathbb{R}$, and $f : \mathbb{R} \longrightarrow \mathbb{R}$ be the function defined by $f(x) = x^5 + px^4 + qx^3 + rx^2 + sx + t$ for any $x \in \mathbb{R}$.

You may take for granted that f is continuous on $\mathbb{R}$.

Define $b = 1 + 2(|p| + |q| + |r| + |s| + |t|)$, and $a = -b$.

i. Prove that $f(b) \geq \frac{b^5}{2}$ and $f(a) \leq -\frac{b^5}{2}$.

ii. Hence apply Bolzano's Intermediate Value Theorem to deduce that f has a zero in (a, b).

(b) Let $g : \mathbb{R} \longrightarrow \mathbb{R}$ be a quintic polynomial function with real coefficients. Prove that g is surjective.

Remark. Can you imitate the argument above to prove that every polynomial function with real coefficient of odd degree is a surjective function from $\mathbb{R}$ to $\mathbb{R}$? What fails to work if you try such an argument on a polynomial function with real coefficient of even degree?

30. *In this question, we are illustrating via specific examples the Cardano-Tartaglia method for finding roots of a general cubic polynomial with complex coefficients.*

Let ω be a cube root of unity. Suppose $\omega \neq 1$.

(a) Verify the 'identities' below, in which each of a, b, c may stand for a complex number or an indeterminate:

i. $a^2 + b^2 + c^2 - ab - bc - ca = (a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$.

ii. $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$.

(b) Consider the polynomial $f(x) = x^3 - 9x + 12$, in which x is the indeterminate.

i. Find real numbers κ, λ which satisfy $\kappa \leq \lambda$ and $\kappa^3 + \lambda^3 = 12$ and $\kappa\lambda = 3$.

ii. Hence, by factorizing $f(x)$, or otherwise, find the roots of $f(x)$ in terms of ω.

Remark. First re-express $f(x)$ in such a way that $\kappa^3, \lambda^3, \kappa\lambda$ appear explicitly.

(c) Consider the polynomial $g(y) = y^3 + 3y^2 - 12y + 10\sqrt{5} - 14$, in which y is the indeterminate.

i. With an appropriate substitution $y = x - \alpha$, in which α is an appropriate constant, re-express $g(y)$ as $x^3 + sx + t$ in which s, t are constants.

ii. Hence find the roots of $g(y)$.

31. Let $\omega = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$. Note that ω is one of the two cube roots of unity which are not 1.

(a) For the moment, take for granted the validity of the statement ($\sharp$) below:

($\sharp$) *Let $h, k \in \mathbb{C}$. There exist some $\sigma, \tau \in \mathbb{C}$ such that $k = \sigma^3 + \tau^3$ and $h = -3\sigma\tau$.*

Let $s, t \in \mathbb{C}$. Consider the polynomial $f(x) = x^3 + sx + t$ with indeterminant x. By factorizing $f(x)$, prove that the polynomial $f(x)$ has a root in $\mathbb{C}$.

Remark. You may need the 'identity' $a^3 + b^3 + c^3 - 3abc = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$.

(b) Let $p, q, r \in \mathbb{C}$. Consider the polynomial $g(y) = y^3 + py^2 + qy + r$ with indeterminate y.

i. Prove that there exists some $\alpha, s, t \in \mathbb{C}$ such that $g(y) = (y - \alpha)^3 + s(y - \alpha) + t$ as polynomials.

ii. Apply the result in the previous part to prove that the polynomial $g(y)$ has a root in $\mathbb{C}$.

(c) Let $G : \mathbb{C} \longrightarrow \mathbb{C}$ be a cubic polynomial function with complex coefficients. Prove that G is surjective.

(d) *We are going to prove the statement ($\sharp$) here.*

Let $h, k \in \mathbb{C}$. Consider the quadratic polynomial $Q(u) = u^2 - ku - \frac{h^3}{27}$ with indeterminate u. Take for granted that the roots of $Q(u)$ in $\mathbb{C}$ are μ, ν respectively, and $Q(u) = (u - \mu)(u - \nu)$ as polynomials.

i. Verify that $\mu + \nu = k$ and $\mu\nu = -\frac{h^3}{27}$.

ii. Hence prove that there exist some $\sigma, \tau \in \mathbb{C}$ such that $\sigma^3 + \tau^3 = k$ and $\sigma\tau = -\frac{h}{3}$.

Remark. Combined together, the results described in this question tell us how we may solve arbitrary cubic polynomial equations with complex coefficients, with the help of the operations $+, -, \times, \div$ and 'taking square roots', 'taking cube roots'. This is the **Cardano-Tartaglia Method**. (But what about quartic polynomial equations? Quintic polynomial equations? You will know the answer from your *algebra* courses.)

32. We introduce/recall these definitions:

- *Let $n \in \mathbb{N}$. A* **degree-n polynomial with complex coefficients and with indeterminate** z *is an expression of the form $a_n z^n + \cdots + a_1 z + a_0$ in which $a_0, a_1, \cdots, a_n \in \mathbb{C}$ and $a_n \neq 0$.*
- *A complex-valued function of one complex variable is called a* **degree-n polynomial function on $\mathbb{C}$** *exactly when its 'formula of definition' is given by a degree-n polynomial with complex coefficients.*
- *Let $\zeta \in \mathbb{C}$, and $f(z) \equiv a_n z^n + \cdots + a_1 z + a_0$ be a polynomial with complex coefficients and with indeterminate z. ζ is said to be a root of the polynomial $f(z)$ in $\mathbb{C}$ if $f(\zeta) = 0$.*

The statement ($\sharp$) below, first proved by Gauss, is known as the **Fundamental Theorem of Algebra**:

($\sharp$) *Every non-constant polynomial with complex coefficients (and with one indeterminate) has a root in $\mathbb{C}$.*

(a) Prove that the statement ($\sharp$) is logically equivalent to the statement ($\flat$) below:

($\flat$) *For any $n \in \mathbb{N}\backslash\{0\}$, every degree-$n$ polynomial function on $\mathbb{C}$ is surjective.*

(b) Let $n \in \mathbb{N}\backslash\{0, 1\}$, $a_0, a_1, \cdots, a_n \in \mathbb{C}$, with $a_n \neq 0$, and $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the degree-n polynomial function defined by $f(z) = a_n z^n + \cdots + a_1 z + a_0$ for any $z \in \mathbb{C}$.

Apply the Fundamental Theorem of Algebra, or otherwise, to prove that f is not injective.

Remark. Here you may also take for granted the **Factor Theorem** (whose 'real version' you have already learnt at school and may be carried in verbatim to the 'complex situation'):

- *Let $\alpha \in \mathbb{C}$, and $p(z)$ be a degree-n polynomial (with complex coefficients). Suppose α is a root of $p(z)$. Then there is a degree-$(n-1)$ polynomial $q(z)$ (with complex coefficients) so that $p(z) = (z - \alpha)q(z)$ as polynomials.*