MATH1050 Characterization of surjectivity with image sets, pre-image sets

1. The statement ($\sharp$) is true:

   ($\sharp$) *Let A, B be sets and $f : A \longrightarrow B$ be a function. For any subset U of B, $f(f^{-1}(U)) \subset U$.*

   Proof of the statement ($\sharp$)?

   - Let A, B be sets and $f : A \longrightarrow B$ be a function. Let U be a subset of B.
     * Pick any object y. Suppose $y \in f(f^{-1}(U))$. Then there exists some $x \in f^{-1}(U)$ such that $y = f(x)$. Since $x \in f^{-1}(U)$, there exists some $z \in U$ such that $z = f(x)$. Now $y = f(x) = z \in U$.

     It follows that $f(f^{-1}(U)) \subset U$.

2. The statement ($\flat$) is false:

   ($\flat$) *Let A, B be sets and $f : A \longrightarrow B$ be a function. For any subset U of B, $f(f^{-1}(U)) \supset U$.*

   Dis-proof of the statement ($\flat$)?

   - Take $A = \{0\}$, $B = \{1, 2\}$, $U = \{1, 2\}$. $U \subset B$. Define the function $f : A \longrightarrow B$ by $f(0) = 1$. $f^{-1}(U) = \{0\}$. $f(f^{-1}(U)) = \{1\}$. We have $2 \in U$ and $2 \notin f(f^{-1}(U))$. Then $U \not\subset f(f^{-1}(U))$.

3. Follow-up questions:

   (a) Can we impose further assumption on f to make the conclusion in the statement ($\flat$) hold? (We are looking for some sufficient condition(s) for the statement ($\flat$).)

   To conceive such an assumption out of nothing may be difficult. Hence we postpone this question.

   (b) What must happen to f if the conclusion in the statement ($\flat$) is true? (We are looking for some necessary condition(s) for the statement ($\flat$).)

   - Suppose that for any subset U of B, $f(f^{-1}(U)) \supset U$. [So what happens?]
     Note that B is a subset of B. Then $f(f^{-1}(B)) \supset B$.
     Now also note that $A = f^{-1}(B)$. Then $f(A) \supset B$.
     Therefore $f(A) = B$. Hence f is surjective.

   (c) Is the necessary condition sufficient?

   - Suppose that f is surjective.
     Let U be a subset of B. [We want to deduce that for any object y, if $y \in U$ then $y \in f(f^{-1}(U))$.]
     * Pick any object y. Suppose $y \in U$. [We want to deduce that $y \in f(f^{-1}(U))$.]
       Since f is surjective, there exists some $x \in A$ such that $y = f(x)$.
       Since $y = f(x)$ and $y \in U$, we have $x \in f^{-1}(U)$.
       Since $y = f(x)$ and $x \in f^{-1}(U)$, we have $y \in f(f^{-1}(U))$.

     It follows that $f(f^{-1}(U)) \supset U$.

4. Conclusion in this investigation? The statement ($\star$) holds:

   ($\star$) *Let A, B be sets and $f : A \longrightarrow B$ be a function. The following statements are equivalent:*

   ($\star_1$) *f is surjective.*

   ($\star_2$) *For any subset U of B, $f(f^{-1}(U)) \supset U$.*

   ($\star_3$) *For any subset U of B, $f(f^{-1}(U)) = U$.*