MATH1050 Theoretical results involving image sets and pre-image sets

1. **Definitions.**
   *Let $A, B$ be sets and $f : A \longrightarrow B$ be a function from $A$ to $B$.*
   (a) *Let $S$ be a subset of $A$. The* **image set of the set $S$ under the function $f$** *is defined to be the set*
   $$\{y \in B : \text{There exists some } x \in S \text{ such that } y = f(x)\}.$$
   *It is denoted by $f(S)$.*
   (b) *Let $U$ be a subset of $B$. The* **pre-image set of the set $U$ under the function $f$** *is defined to be the set*
   $$\{x \in A : \text{There exists some } y \in U \text{ such that } y = f(x)\}.$$
   *It is denoted by $f^{-1}(U)$.*

2. **Theorem (1).**
   *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:*
   (1a) $f(\emptyset) = \emptyset$.
   (1b) $f^{-1}(\emptyset) = \emptyset$.
   (1c) $f(A) \subset B$.
   (1d) $f^{-1}(B) = A$.
   (1e) *Let $x \in A$.* $f(\{x\}) = \{f(x)\}$.
   (1f) *Let $x \in A$, $y \in B$. The statements below are logically equivalent:*
   (i) $x \in f^{-1}(\{y\})$.
   (ii) $f(x) \in \{y\}$.
   (iii) $f(x) = y$.

   The proof of Theorem (1) is left as an exercise.

3. **Theorem (2).**
   *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:*
   (2a) *Let $S, T$ be subsets of $A$. Suppose $S \subset T$. Then $f(S) \subset f(T)$.*
   (2b) *Let $H, K$ be subsets of $A$.*
   (1) $f(H \cup K) \supset f(H) \cup f(K)$.
   (2) $f(H \cup K) \subset f(H) \cup f(K)$.
   (3) $f(H \cup K) = f(H) \cup f(K)$.
   (2c) *Let $H, K$ be subsets of $A$.* $f(H \cap K) \subset f(H) \cap f(K)$.

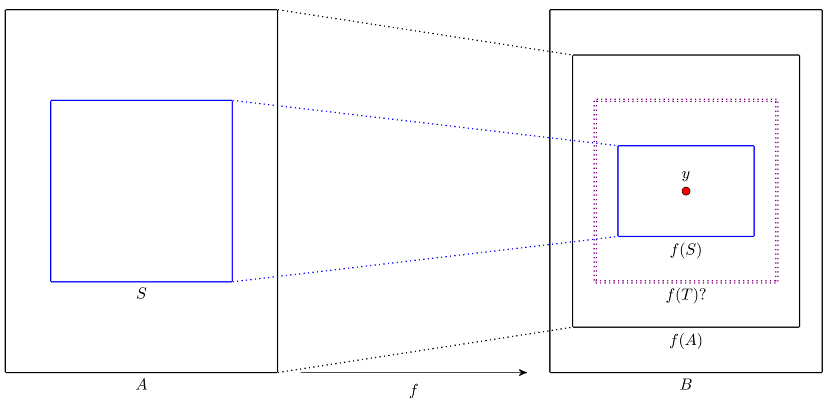
   We give the proof of Theorem (2) below.

4. **Proof of Statement (2a) of Theorem (2) (with pictures).**
   Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. Let $S, T$ be subsets of $A$. Suppose $S \subset T$. [We want to deduce that $f(S) \subset f(T)$. What to do, really? We want to prove:

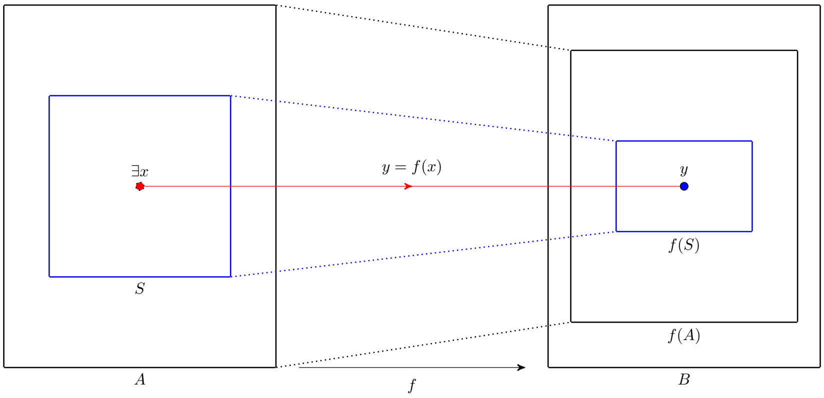
   'for any object $y$, if $y \in f(S)$ then $y \in f(T)$.'

   Think about this before proceeding any further.]

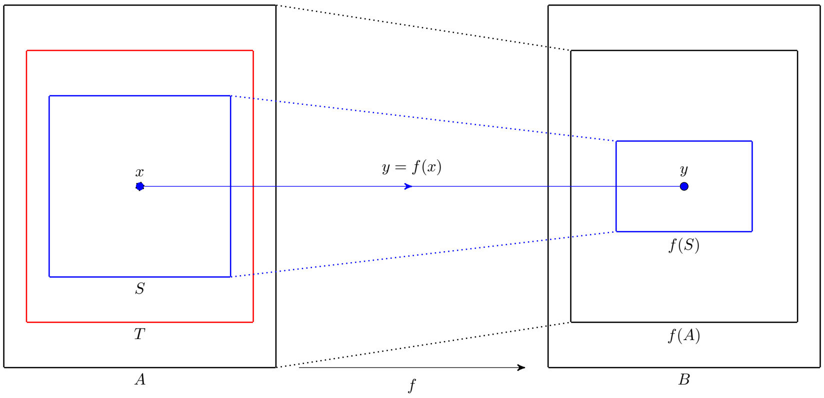
   (1) Pick any object $y$. [From now on, this $y$ is fixed.]
   Suppose $y \in f(S)$. [We want to deduce that $y \in f(T)$.]



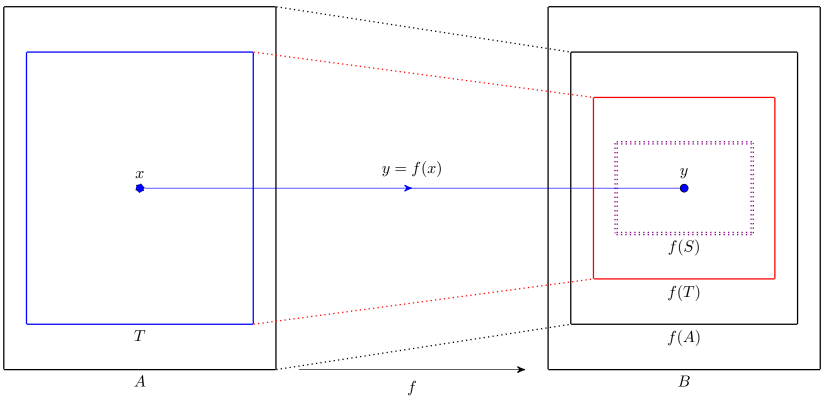
(2) Then [according to the definition of image sets,] there exists some $x \in S$ such that $y = f(x)$. [What does it say about $x, y$?]




(3) Since $x \in S$ and $S \subset T$, we have $x \in T$ [according to the definition of subsets].




(4) For the same $x, y$, since $x \in T$ and $y = f(x)$, we have $y \in f(T)$ [according to the definition of image sets].




It follows that $f(S) \subset f(T)$.

**Proof of Statement (2a) of Theorem (2) (without pictures).**

Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. Let $S, T$ be subsets of $A$. Suppose $S \subset T$. [We want to deduce that $f(S) \subset f(T)$.]

Pick any object $y$.
Suppose $y \in f(S)$. [We want to deduce that $y \in f(T)$.]
Then there exists some $x \in S$ such that $y = f(x)$.
Since $x \in S$ and $S \subset T$, we have $x \in T$.
For the same $x, y$, since $x \in T$ and $y = f(x)$, we have $y \in f(T)$.

It follows that $f(S) \subset f(T)$.

**Very formal proof of Statement (2a) of Theorem (2).**

Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. Let $S, T$ be subsets of $A$. Suppose $S \subset T$. [We want to deduce that $f(S) \subset f(T)$.] Pick any object $y$.

- [We want to prove that if $y \in f(S)$ then $y \in f(T)$.]

  **I**. Suppose $y \in f(S)$. [Assumption.]

  **II**. $S \subset T$. [Assumption.]

  **III**. There exists some $x \in S$ such that $y = f(x)$. [**I**, definition of image sets.]

  **IV**. $x \in S$. [**III**.]

  **V**. $x \in T$. [**II**, **IV**, definition of subsets.]

  **VI**. $y = f(x)$. [**III**.]

  **VII**. $y \in f(T)$. [**V**, **VI**, definition of image sets.]

It follows that $f(S) \subset f(T)$.

5. **Proof of Statement (2b) of Theorem (2).**

   Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. Let $H, K$ be subsets of $A$.

   (1) [We want to prove that $f(H) \cup f(K) \subset f(H \cup K)$.]
   By (2a), since $H \subset H \cup K$, we have $f(H) \subset f(H \cup K)$.
   Also by (2a), since $K \subset H \cup K$, we have $f(K) \subset f(H \cup K)$.
   Then $f(H) \cup f(K) \subset f(H \cup K)$. [Why?]

   (2) [We want to prove that $f(H \cup K) \subset f(H) \cup f(K)$. What does it amount to? Focus on '$\subset$'.
   We want to prove:

   'for any object $y$, if $y \in f(H \cup K)$ then $y \in f(H) \cup f(K)$.'

   Think about it before proceeding any further.]
   Pick any object $y$. Suppose $y \in f(H \cup K)$. [We want to deduce that $y \in f(H) \cup f(K)$.]
   [Now make use of the definition of image sets.] There exists some $x \in H \cup K$ such that $y = f(x)$. [What does it say about $x, y$?]
   Since $x \in H \cup K$, we have $x \in H$ or $x \in K$.

   * (Case 1). Suppose $x \in H$. Since $y = f(x)$ and $x \in H$, we have $y \in f(H)$. Then $y \in f(H)$ or $y \in f(K)$.
   * (Case 2). Suppose $x \in K$. Modifying the argument for (Case 1), we also deduce that $y \in f(H)$ or $y \in f(K)$.

   Hence, in any cases, we have $y \in f(H) \cup f(K)$. It follows that $f(H \cup K) \subset f(H) \cup f(K)$.

   (3) By (2b1), (2b2), we have $f(H \cup K) = f(H) \cup f(K)$.

6. **Proof of Statement (2c) of Theorem (2).**

   Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. Let $H, K$ be subsets of $A$. [We want to prove $f(H \cap K) \subset f(H) \cap f(K)$. Focus on '$\subset$'.]

   - [We apply (2a).]
   Since $H \cap K \subset H$, we have $f(H \cap K) \subset f(H)$.
   Since $H \cap K \subset K$, we have $f(H \cap K) \subset f(K)$.
   Then $f(H \cap K) \subset f(H) \cap f(K)$.

7. **Theorem (3).**

   *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:*

   (3a) *Let $U, V$ be subsets of $B$. Suppose $U \subset V$. Then $f^{-1}(U) \subset f^{-1}(V)$.*

   (3b) *Let $U, V$ be subsets of $B$.*

   (1) $f^{-1}(U \cup V) \supset f^{-1}(U) \cup f^{-1}(V)$.
   (2) $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.
   (3) $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

   (3c) *Let $U, V$ subsets of $B$.*

   (1) $f^{-1}(U \cap V) \subset f^{-1}(U) \cap f^{-1}(V)$.
   (2) $f^{-1}(U \cap V) \supset f^{-1}(U) \cap f^{-1}(V)$.
   (3) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.

   We give the proof of Statement (3b2) of Theorem (3) below. The proof of the rest of Theorem (3) is left as exercises.

8. **Proof of Statement (3b2) of Theorem (3).**

   Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Let $U, V$ be subsets of $B$.

   [We want to prove that $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.

   What to do, really? We want to prove:

   'for any object $x$, if $x \in f^{-1}(U \cup V)$ then $x \in f^{-1}(U) \cup f^{-1}(V)$.'

   Think about this before proceeding any further.]

   Pick any object $x$.
   Suppose $x \in f^{-1}(U \cup V)$. Then [according to the definition of pre-image sets,] there exists some $y \in U \cup V$ such that $y = f(x)$.
   Since $y \in U \cup V$, we have $y \in U$ or $y \in V$ [according to the definition of unions].

   * (Case 1). Suppose $y \in U$. [Recall $y = f(x)$.] Then $y = f(x)$ for this $y \in U$. Therefore $x \in f^{-1}(U)$ [according to the definition of pre-image sets].
   * (Case 2). Suppose $y \in V$. [Recall $y = f(x)$.] Then $y = f(x)$ for this $y \in V$. Therefore $x \in f^{-1}(V)$ [according to the definition of pre-image sets].

   Now $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$. Therefore $x \in f^{-1}(U) \cup f^{-1}(V)$ [according to the definition of unions].

   It follows that $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.

9. **Remark.**

   Which of the statements is true? Which not?

   (a) *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Let $S, T$ be subsets of $A$. Suppose $f(S) \subset f(T)$. Then $S \subset T$.*
   (b) *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Let $U, V$ be subsets of $B$. Suppose $f^{-1}(U) \subset f^{-1}(V)$. Then $U \subset V$.*
   (c) *Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. Let $H, K$ be subsets of $A$. $f(H \cap K) \supset f(H) \cap f(K)$.*

   They are all false. (Can you provide counter-examples for the respective dis-proofs?)

10. **Theorem (4).**

*Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. The following statements hold:*

(4a) *Let $S$ be a subset of $A$. $(g \circ f)(S) = g(f(S))$.*

(4b) *Let $W$ be a subset of $C$. $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.*

We give the proof of Statement (4b) below; the proof of Statement (4a) is left as an exercise.

11. **Proof of Statement (4b) of Theorem (4).**

Let $A, B, C$ be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Let $W$ be a subset of $C$.

[We want to prove the set equality $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.

Hence we separate arguments into two parts, each on a 'subset relation'.

Which two?

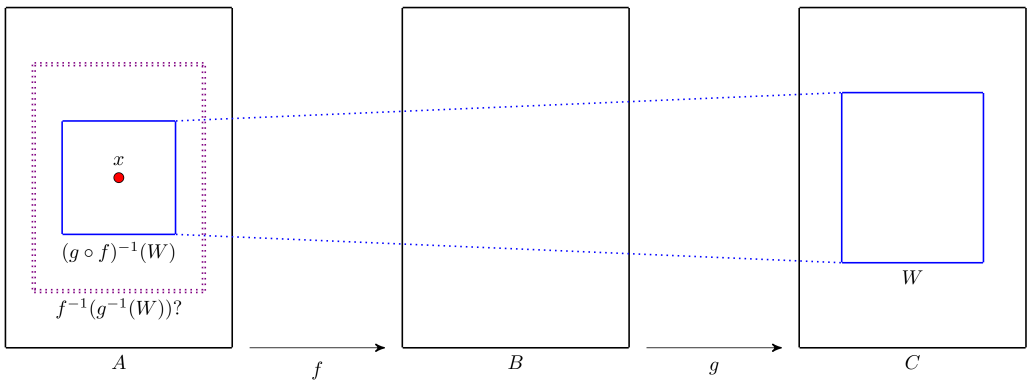
($\alpha$) $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.
This reads:
'for any object $x$, if $x \in (g \circ f)^{-1}(W)$ then $x \in f^{-1}(g^{-1}(W))$.'

($\beta$) $(g \circ f)^{-1}(W) \supset f^{-1}(g^{-1}(W))$.
This reads:
'for any object $x$, if $x \in f^{-1}(g^{-1}(W))$ then $x \in (g \circ f)^{-1}(W)$.'

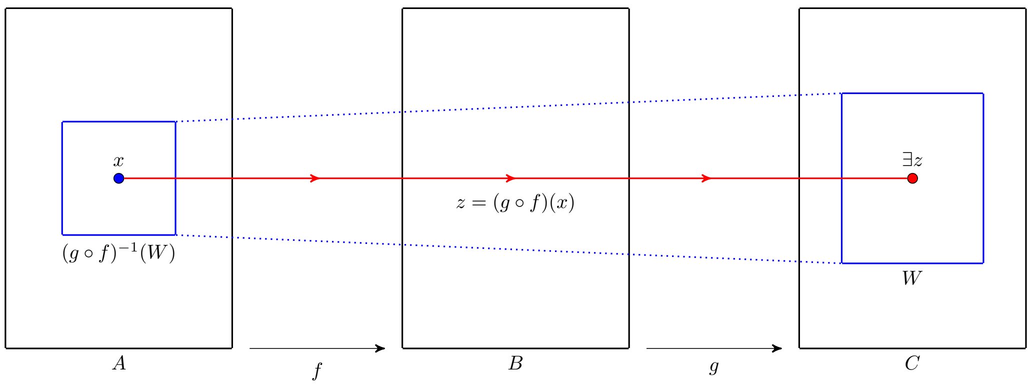
Think about this before proceeding any further.]

($\alpha$) [We are going to prove that $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.]

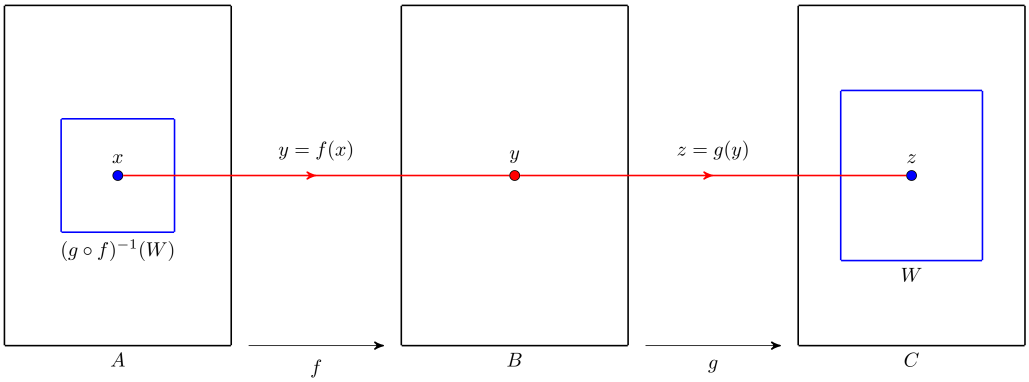
(1) Pick any object $x$.
Suppose that $x \in (g \circ f)^{-1}(W)$. [We want to deduce that $x \in f^{-1}(g^{-1}(W))$.]



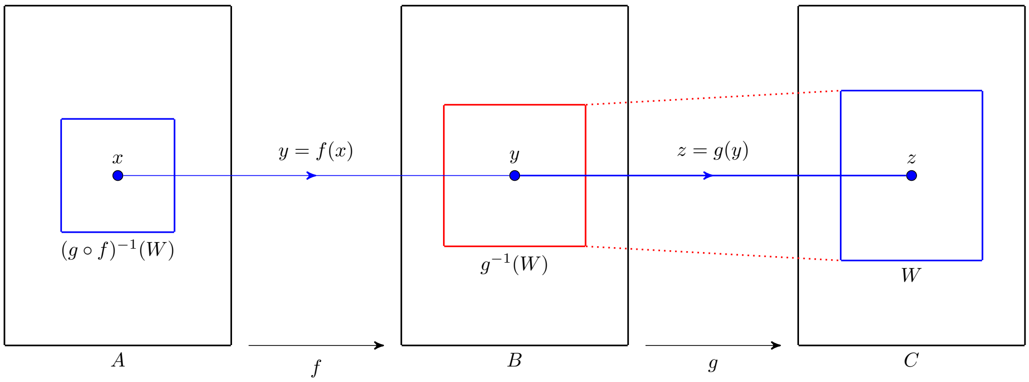

(2) Then [according to the definition of pre-image sets,] there exists some $z \in W$ such that $z = (g \circ f)(x)$.



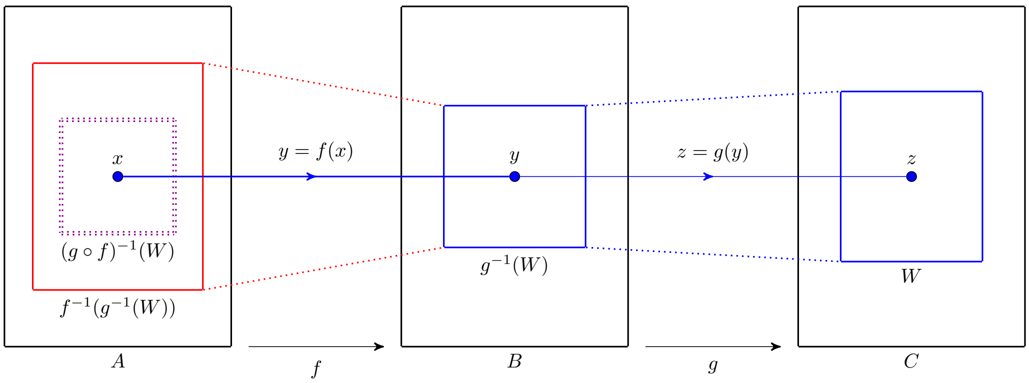
(3) We have $z = g(f(x))$ [by the definition of composition of functions].
Take $y = f(x)$. Note that $y \in B$.




(4) We have $g(y) = g(f(x)) = z$ and $z \in W$. Then $y \in g^{-1}(W)$ [according to the definition of pre-image sets].




(5) $f(x) = y$ and $y \in g^{-1}(W)$. Then $x \in f^{-1}(g^{-1}(W))$ [according to the definition of pre-image sets].




($\beta$) [We are going to prove that $(g \circ f)^{-1}(W) \supset f^{-1}(g^{-1}(W))$.]
Pick any object $x$. Suppose $x \in f^{-1}(g^{-1}(W))$. [We want to deduce that $x \in (g \circ f)^{-1}(W)$.]
Then [according to the definition of pre-image sets,] there exists some $y \in g^{-1}(W)$ such that $y = f(x)$.
Now $y \in g^{-1}(W)$. Then [according to the definition of pre-image sets,] there exists some $z \in W$ such that $z = g(y)$.
We have $z = g(y) = g(f(x)) = (g \circ f)(x)$ and $z \in W$. Then [according to the definition of pre-image sets,] we have $x \in (g \circ f)^{-1}(W)$.

It follows that $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.

12. **Theorem (5).**
*Let $A, B$ be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:*

(5a) *Let $S$ be a subset of $A$. $f^{-1}(f(S)) \supset S$.*

(5b) *Let $U$ be a subset of $B$. $f(f^{-1}(U)) \subset U$.*

(5c) *Let $S$ be a subset of $A$. $f(f^{-1}(f(S))) = f(S)$.*

(5d) *Let $U$ be a subset of $B$. $f^{-1}(f(f^{-1}(U))) = f^{-1}(U)$.*

(5e) *Let $S$ be a subset of $A$, and $U$ be a subset of $B$. $f(S \cap f^{-1}(U)) = f(S) \cap U$.*

The proof of Theorem (5) is left as an exercise.

13. **Definition.**

*Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. $f$ is said to be* **surjective** *if the statement $(S)$ holds:*

> *$(S)$: For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.*

**Theorem (6). (Characterizations of surjectivity).**

*Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. The following statements are equivalent:*

(I) *$f$ is surjective.*

(Ia) $f(A) = B$.

(Ib) $f(A) \supset B$.

(II) *For any subset $U$ of $B$, $f(f^{-1}(U)) \supset U$.*

(IIa) *For any subset $U$ of $B$, $f(f^{-1}(U)) = U$.*

(III) *For any subset $U$ of $B$, there exists some subset $S$ of $A$ such that $U = f(S)$.*

(IV) *For any subset $T$ of $A$, $f(A \backslash T) \supset B \backslash f(T)$.*

(V) *For any subset $U, V$ of $B$, if $f^{-1}(U) \subset f^{-1}(V)$ then $U \subset V$.*

(VI) *For any subset $U, V$ of $B$, if $f^{-1}(U) = f^{-1}(V)$ then $U = V$.*

**Proof of Theorem (6).** Let $A, B$ be sets and $f : A \longrightarrow B$ be a function.

- The justification for the equivalence of (I), (Ia), (Ib) is a simple 'game of words'. The justification for the equivalence of (II), (IIa) is also a simple 'game of words'. They are left as exercises.
- ['(I) $\Longrightarrow$ (II)'?] Suppose (I) holds. Let $U$ be a subset of $B$. [We want to deduce that for any object $y$, if $y \in U$ then $y \in f(f^{-1}(U))$.]

  Pick any object $y$. Suppose $y \in U$. [We want to deduce that $y \in f(f^{-1}(U))$.] Since $f$ is surjective, there exists some $x \in A$ such that $y = f(x)$. Since $y = f(x)$ and $y \in U$, we have $x \in f^{-1}(U)$. Since $y = f(x)$ and $x \in f^{-1}(U)$, we have $y \in f(f^{-1}(U))$.

  It follows that $f(f^{-1}(U)) \supset U$.
- ['(II) $\Longrightarrow$ (III)'?] Suppose that (II) holds. Let $U$ be a subset of $B$. Take $S = f^{-1}(U)$. $S \subset A$ and $f(S) = f(f^{-1}(U)) = U$.
- ['(III) $\Longrightarrow$ (I)'?] Suppose that (III) holds. Note that $B$ is a subset of $B$. Then there exists some subset $S$ of $A$ such that $B = f(S)$. Since $S \subset A$, we have $B = f(S) \subset f(A)$. Hence $f$ is surjective.
- ['(I) $\Longrightarrow$ (IV)'?] Suppose that (I) holds. Let $T$ be a subset of $A$. Pick any object $y$. Suppose $y \in B \backslash f(T)$. Then $y \in B$ and $y \notin f(T)$. By surjectivity, there exists some $x \in A$ such that $y = f(x)$. We claim that $x \in A \backslash T$:

  > [Justification using proof-by-contradiction.] Suppose it were true that $x \notin A \backslash T$. Then $x \in T$. We would then have $y = f(x) \in f(T)$. But $y \notin f(T)$ in the first place. Contradiction arises.

  Now $x \in A \backslash T$. Therefore $y = f(x) \in f(A \backslash T)$. It follows that $B \backslash f(T) \subset f(A \backslash T)$.
- ['(IV) $\Longrightarrow$ (I)'?] Suppose that (IV) holds. Then $f(A) = f(A \backslash \emptyset) \supset B \backslash f(\emptyset) = B \backslash \emptyset = B$.
- ['(I) $\Longrightarrow$ (V)'?] Exercise.
- ['(V) $\Longrightarrow$ (VI)'?] Exercise: game of words.
- ['(VI) $\Longrightarrow$ (I)'?] Exercise.

14. **Definition.**

*Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. $f$ is said to be* **injective** *if the statement $(I)$ holds:*

$(I)$: *For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.*

**Theorem (7). (Characterizations of injectivity).**

*Let $A, B$ be sets and $f : A \longrightarrow B$ be a function. The following statements are equivalent:*

(I) *$f$ is injective.*

(II) *For any subset $S$ of $A$, $f^{-1}(f(S)) \subset S$.*

(IIa) *For any subset $S$ of $A$, $f^{-1}(f(S)) = S$.*

(III) *For any subset $S$ of $A$, there exists some subset $U$ of $B$ such that $S = f^{-1}(U)$.*

(IV) *For any subset $S, T$ of $A$, $f(S \cap T) \supset f(S) \cap f(T)$.*

(IVa) *For any subset $S, T$ of $A$, $f(S \cap T) = f(S) \cap f(T)$.*

(V) *For any subsets $S, T$ of $A$, if $f(S) \subset f(T)$ then $S \subset T$.*

(VI) *For any subsets $S, T$ of $A$, if $f(S) = f(T)$ then $S = T$.*

The proof of Theorem (7) is left as an exercise.

15. **Theorem (8).**

*Let $A, B$ be sets and $f : A \longrightarrow B$ be a function.*

(8a) *Let $\{U_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of $B$. ($\{f^{-1}(U_n)\}_{n=0}^{\infty}$ is an infinite sequence of subsets of $A$.) The following statements hold:*

(1) $f^{-1}(\bigcap_{n=0}^{\infty} U_n) = \bigcap_{n=0}^{\infty} f^{-1}(U_n)$.

(2) $f^{-1}(\bigcup_{n=0}^{\infty} U_n) = \bigcup_{n=0}^{\infty} f^{-1}(U_n)$.

(8b) *Let $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of $A$. ($\{f(S_n)\}_{n=0}^{\infty}$ is an infinite sequence of subsets of $B$.) The following statements hold:*

(1) $f(\bigcap_{n=0}^{\infty} S_n) \subset \bigcap_{n=0}^{\infty} f(S_n)$.

(2) $f(\bigcup_{n=0}^{\infty} S_n) = \bigcup_{n=0}^{\infty} f(S_n)$.

(8c) *The statements below are logically equivalent:*

(i) *$f$ is injective.*

(ii) *For any infinite sequence of subsets $\{S_n\}_{n=0}^{\infty}$ of $A$, $f(\bigcap_{n=0}^{\infty} S_n) = \bigcap_{n=0}^{\infty} f(S_n)$.*

The proof of Theorem (8) is left as an exercise (in quantifiers).