1. **Definitions**.

Let A, B be sets and $f : A \longrightarrow B$ be a function from A to B.

(a) *Let S be a subset of A. The* **image set of the set S under the function f** *is defined to be the set*

$$\{y \in B : \text{There exists some } x \in S \text{ such that } y = f(x)\}.$$

It is denoted by $f(S)$.

(b) *Let U be a subset of B. The* **pre-image set of the set U under the function f** *is defined to be the set*

$$\{x \in A : \text{There exists some } y \in U \text{ such that } y = f(x)\}.$$

It is denoted by $f^{-1}(U)$.

2. **Theorem (1)**.

Let A, B be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:

(1a) $f(\emptyset) = \emptyset$.

(1b) $f^{-1}(\emptyset) = \emptyset$.

(1c) $f(A) \subset B$.

(1d) $f^{-1}(B) = A$.

(1b) *Let $x \in A$. $f(\{x\}) = \{f(x)\}$.*

(1c) *Let $x \in A$, $y \in B$. The statements below are logically equivalent:*

(i) $x \in f^{-1}(\{y\})$.

(ii) $f(x) \in \{y\}$.

(iii) $f(x) = y$.

3. **Theorem (2).**

Let A, B be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:

(2a) *Let S, T be subsets of A. Suppose $S \subset T$. Then $f(S) \subset f(T)$.*

(2b) *Let H, K be subsets of A.*

(1) $f(H \cup K) \supset f(H) \cup f(K)$.

(2) $f(H \cup K) \subset f(H) \cup f(K)$.

(3) $f(H \cup K) = f(H) \cup f(K)$.

(2c) *Let H, K be subsets of A. $f(H \cap K) \subset f(H) \cap f(K)$.*

4. **Proof of Statement (2a) of Theorem (2)** .

Let A, B be sets and $f : A \longrightarrow B$ be a function.

Let S, T be subsets of A. Suppose $S \subset T$.

[We want to deduce that $f(S) \subset f(T)$.

What to do, really? We want to prove:

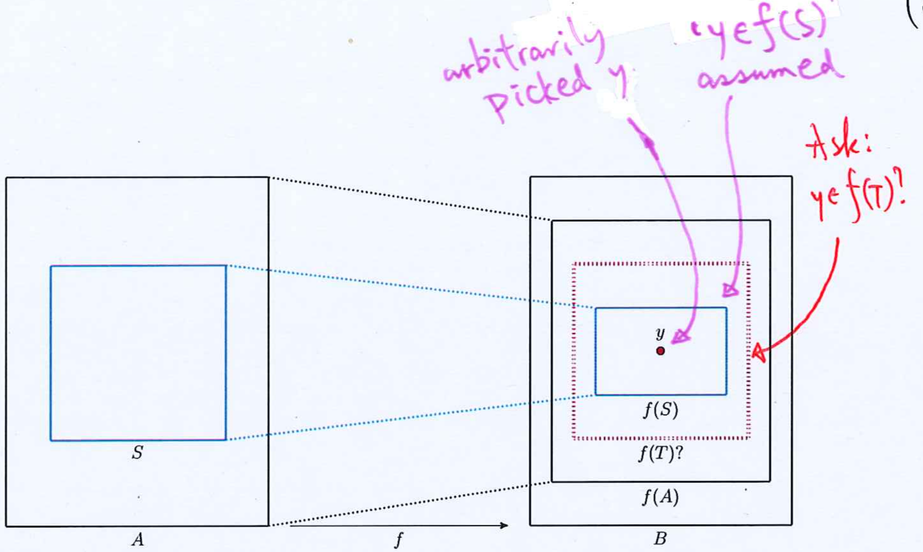
'For any object y, if $y \in f(S)$ then $y \in f(T)$.'

Think about this before proceeding any further.]

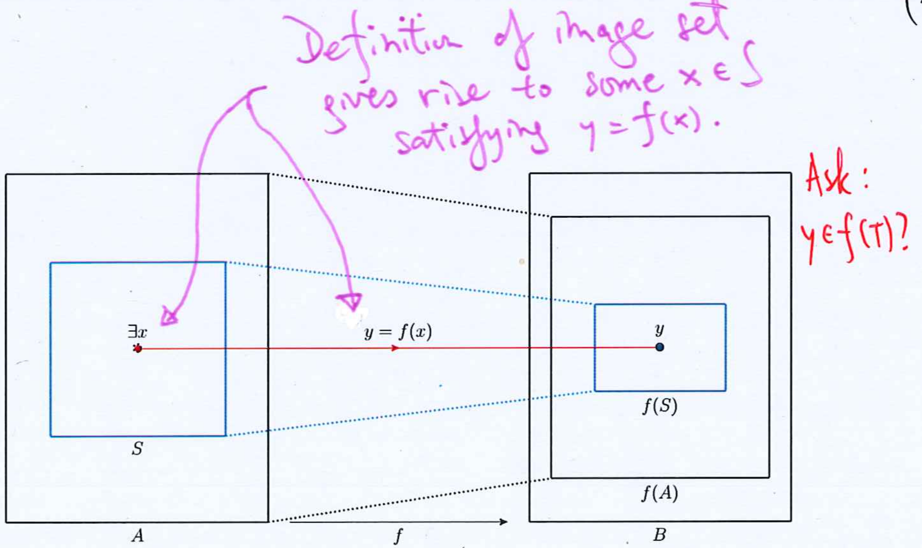
Proof of Statement (2a) of Theorem (2).

... Suppose $S \subset T$. [Want to prove: $f(S) \subset f(T)$. This reads: 'for any y, if $y \in f(S)$ then $y \in f(T)$'.]

(1)



(2)



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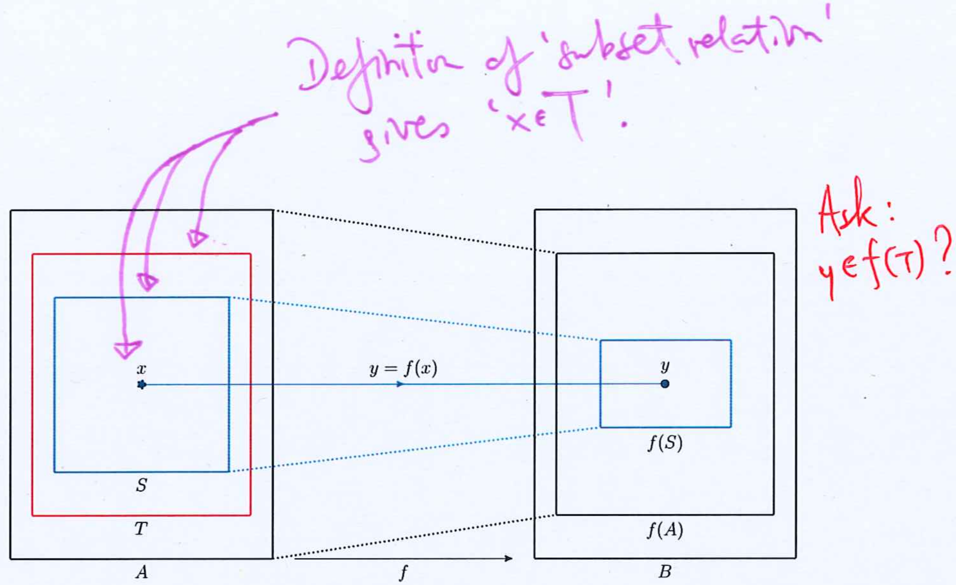

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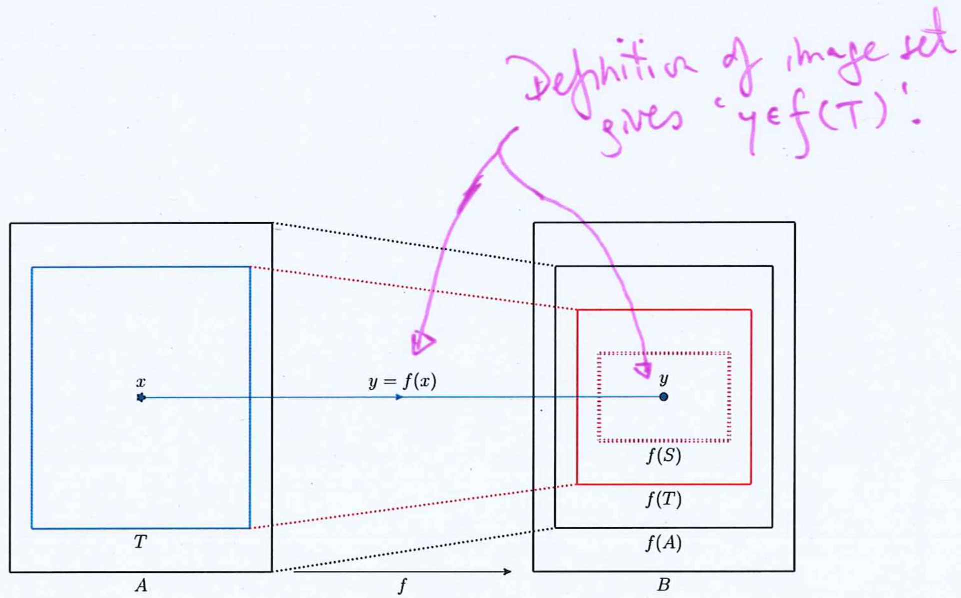
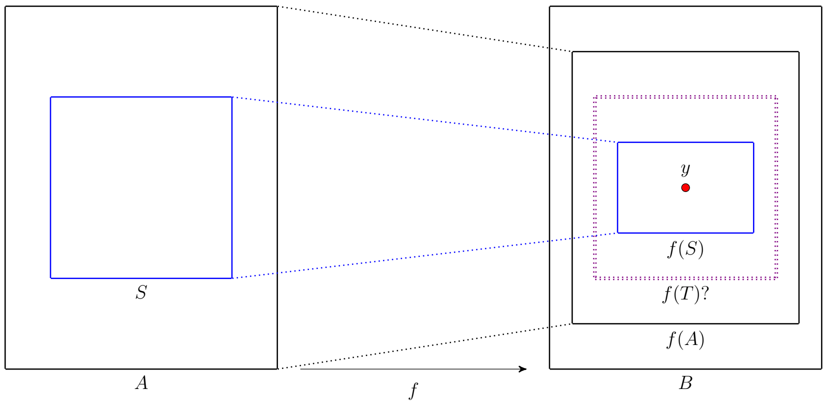


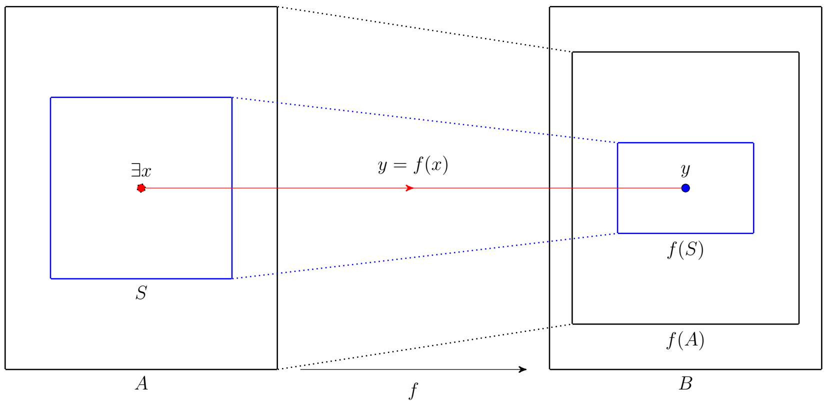
Proof of Statement (2a) of Theorem (2) .

... Suppose $S \subset T$. [Want to prove: $f(S) \subset f(T)$. This reads: 'for any y, if $y \in f(S)$ then $y \in f(T)$'.]

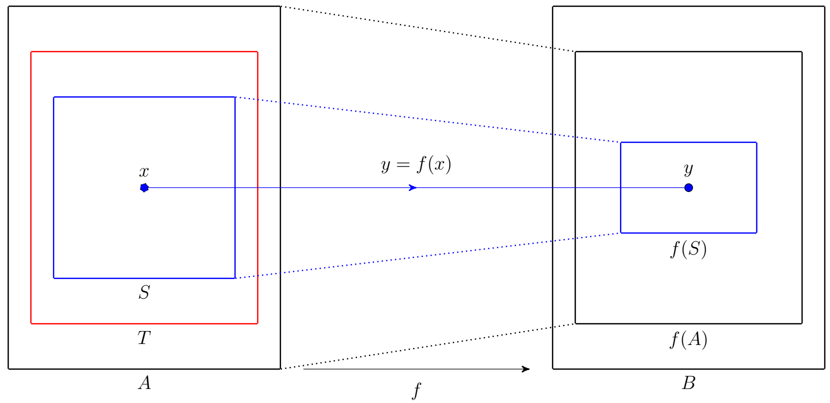
(1) Pick any object y. [From now on, this y is fixed.]
Suppose $y \in f(S)$. [Want to deduce: $y \in f(T)$.]



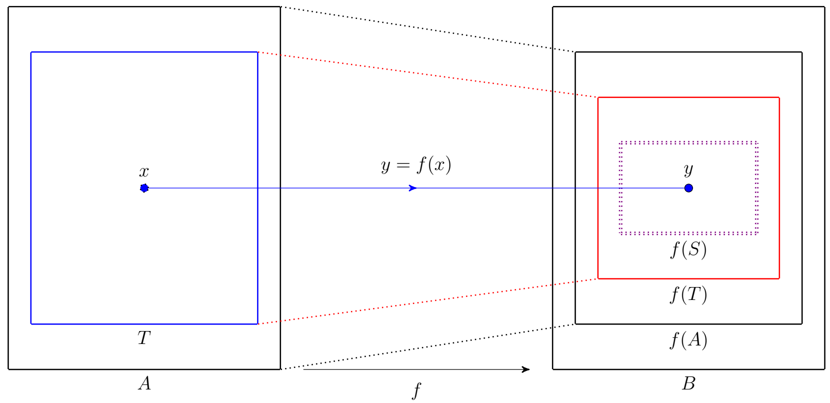
(2) Then there exists some $x \in S$ such that $y = f(x)$. [From now on, this x is fixed.]
[Reason: Definition of image sets.]



(3) Since $x \in S$ and $S \subset T$, we have $x \in T$.
[Reason: Definition of notion of subset.]



(4) For the same x, y,
since $x \in T$ and $y = f(x)$, we have $y \in f(T)$.
[Reason: Definition of image sets.]



It follows that $f(S) \subset f(T)$. $\square$

5. **Proof of Statement (2b) of Theorem (2).**

Let A, B be sets and $f : A \longrightarrow B$ be a function. Let H, K be subsets of A.

(1) [We want to prove that $f(H) \cup f(K) \subset f(H \cup K)$.]

Note that $H \subset H \cup K$. Then, by (2a), we have $f(H) \subset f(H \cup K)$.
Also note that $K \subset H \cup K$. Then, by (2a), we have $f(K) \subset f(H \cup K)$.
Since $f(H) \subset f(H \cup K)$ and $f(K) \subset f(H \cup K)$, we have $f(H) \cup f(K) \subset f(H \cup K)$.

(2) [We want to prove that $f(H \cup K) \subset f(H) \cup f(K)$.

What to do, really? We want to prove:

'For any object y, if $y \in f(H \cup K)$ then $y \in f(H) \cup f(K)$.'

Think about this before proceeding any further.]

Pick any object y. Suppose $y \in f(H \cup K)$.
Then, by the definition of image sets,
there exists some $x \in H \cup K$ such that $y = f(x)$.
Since $x \in H \cup K$, we have $x \in H$ or $x \in K$.

* (Case 1). Suppose $x \in H$.
Since $x \in H$ and $y = f(x)$, we have $y \in f(H)$.
Then $y \in f(H)$ or $y \in f(K)$. Therefore $y \in f(H) \cup f(K)$.

* (Case 2). Suppose $x \in K$.
[...]
Therefore $y \in f(H) \cup f(K)$.

Hence, in any case, $y \in f(H) \cup f(K)$.

It follows that $f(H \cup K) \subset f(H) \cup f(K)$. □

(3) By (2b1), (2b2), we have $f(H \cup K) = f(H) \cup f(K)$.

6. **Proof of Statement (2c) of Theorem (2).**

Let A, B be sets and $f : A \longrightarrow B$ be a function. Let H, K be subsets of A.

[We want to prove $f(H \cap K) \subset f(H) \cap f(K)$.]

- Note that $H \cap K \subset H$.
  Then, by (2a), we have $f(H \cap K) \subset f(H)$.
  Also note that $H \cap K \subset K$.
  Then, by (2a), we have $f(H \cap K) \subset f(K)$.
  Since $f(H \cap K) \subset f(H)$ and $f(H \cap K) \subset f(K)$,
  we have $f(H \cap K) \subset f(H) \cap f(K)$. $\square$

7. **Theorem (3).**

Let A, B be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:

(3a) *Let U, V be subsets of B. Suppose $U \subset V$. Then $f^{-1}(U) \subset f^{-1}(V)$.*

(3b) *Let U, V be subsets of B.*

(1) $f^{-1}(U \cup V) \supset f^{-1}(U) \cup f^{-1}(V)$.

(2) $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.

(3) $f^{-1}(U \cup V) = f^{-1}(U) \cup f^{-1}(V)$.

(3c) *Let U, V subsets of B.*

(1) $f^{-1}(U \cap V) \subset f^{-1}(U) \cap f^{-1}(V)$.

(2) $f^{-1}(U \cap V) \supset f^{-1}(U) \cap f^{-1}(V)$.

(3) $f^{-1}(U \cap V) = f^{-1}(U) \cap f^{-1}(V)$.

8. **Proof of Statement (3b2) of Theorem (3).**

Let A, B be sets, and $f : A \longrightarrow B$ be a function. Let U, V be subsets of B.

[We want to prove that $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.

What to do, really? We want to prove:

'For any object x, if $x \in f^{-1}(U \cup V)$ then $x \in f^{-1}(U) \cup f^{-1}(V)$.'

Think about this before proceeding any further.]

Proof of Statement (3b2) of Theorem (3).

... Let U, V be subsets of B.

[Want to prove: $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$.
This reads: 'For any object x, if $x \in f^{-1}(U \cup V)$ then $x \in f^{-1}(U) \cup f^{-1}(V)$'.]

Pick any object x.

Suppose $x \in f^{-1}(U \cup V)$.

Then, by the definition of pre-image sets, there exists some $y \in U \cup V$ such that $y = f(x)$.

Since $y \in U \cup V$, we have $y \in U$ or $y \in V$.

* (Case 1). Suppose $y \in U$.
  Since $y \in U$ and $y = f(x)$, we have $x \in f^{-1}(U)$.
  Then $x \in f^{-1}(U)$ or $x \in f^{-1}(V)$.
  Therefore $x \in f^{-1}(U) \cup f^{-1}(V)$.
* (Case 2). Suppose $y \in V$. [...] Therefore $x \in f^{-1}(U) \cup f^{-1}(V)$.

Hence, in any case, we have $x \in f^{-1}(U) \cup f^{-1}(V)$.

It follows that $f^{-1}(U \cup V) \subset f^{-1}(U) \cup f^{-1}(V)$. $\square$

9. **Remark.**

Which of the statements is true? Which not?

(a) *Let A, B be sets, and $f : A \longrightarrow B$ be a function.*

Let S, T be subsets of A.

Suppose $f(S) \subset f(T)$. Then $S \subset T$.

(b) *Let A, B be sets, and $f : A \longrightarrow B$ be a function.*

Let U, V be subsets of B.

Suppose $f^{-1}(U) \subset f^{-1}(V)$. Then $U \subset V$.

(c) *Let A, B be sets, and $f : A \longrightarrow B$ be a function.*

Let H, K be subsets of A.

$f(H \cap K) \supset f(H) \cap f(K)$.

They are all false. (Can you provide counter-examples for the respective dis-proofs?)

10. **Theorem (4).**

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. The following statements hold:

(4a) *Let S be a subset of A. $(g \circ f)(S) = g(f(S))$.*

(4b) *Let W be a subset of C. $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.*

11. **Proof of Statement (4b) of Theorem (4).**

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Let W be a subset of C.

[We want to prove the set equality $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.

Hence we separate arguments into two parts, each on a 'subset relation'.

Which two?

(α) $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.

This reads: 'For any object x, if $x \in (g \circ f)^{-1}(W)$ then $x \in f^{-1}(g^{-1}(W))$.'

(β) $(g \circ f)^{-1}(W) \supset f^{-1}(g^{-1}(W))$.

This reads: 'For any object x, if $x \in f^{-1}(g^{-1}(W))$ then $x \in (g \circ f)^{-1}(W)$.'

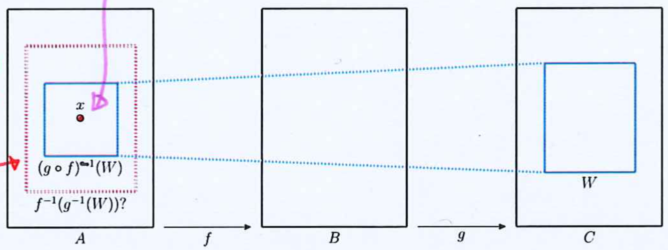
Think about this before proceeding any further.]

(α) [We are going to prove that $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.] [For any x, if $x \in (g \circ f)^{-1}(W)$ then $x \in f^{-1}(g^{-1}(W))$.]

(1)

arbitrarily picked x.
'$x \in (g \circ f)^{-1}(W)$' assumed.

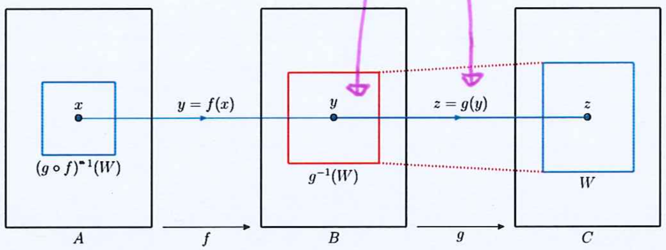
Ask: $x \in f^{-1}(g^{-1}(W))$?



(4)

Definition of pre-image sets gives $y \in g^{-1}(W)$.

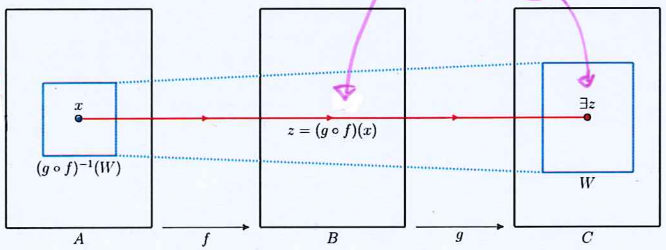
Ask: $x \in f^{-1}(g^{-1}(W))$?



(2)

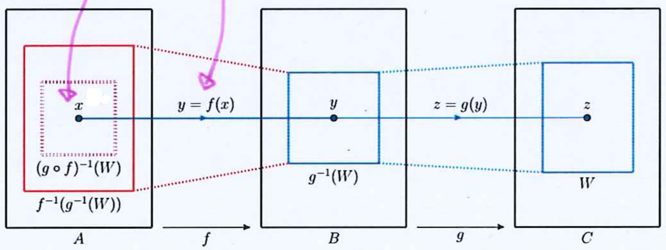
Definition of pre-image sets gives rise to some $z \in W$ satisfying $z = (g \circ f)(x)$.

Ask: $x \in f^{-1}(g^{-1}(W))$?



(5)

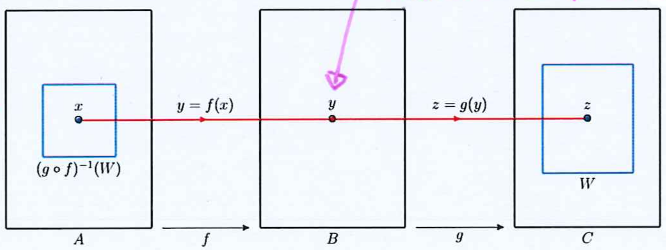
Definition of pre-image sets gives $x \in f^{-1}(g^{-1}(W))$.



(3)

Definition of composition gives $z = g(f(x))$.
Introduce $y = f(x)$.

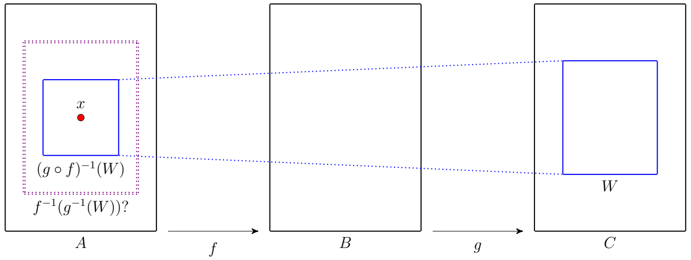
Ask: $x \in f^{-1}(g^{-1}(W))$?



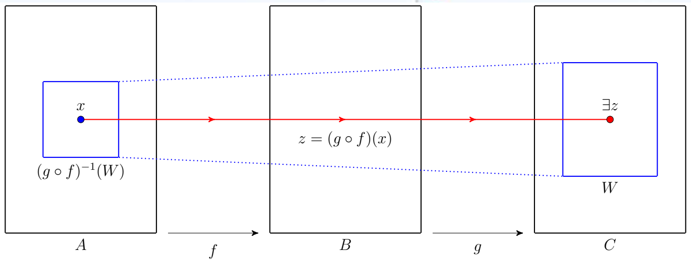
(α) [We are going to prove that $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.]

[For any x, if $x \in (g \circ f)^{-1}(W)$ then $x \in f^{-1}(g^{-1}(W))$.]

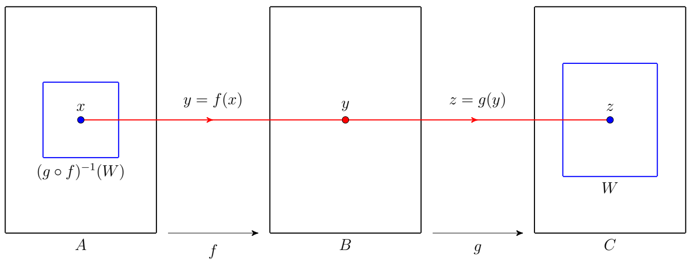
(1) Pick any object x.
Suppose $x \in (g \circ f)^{-1}(W)$.



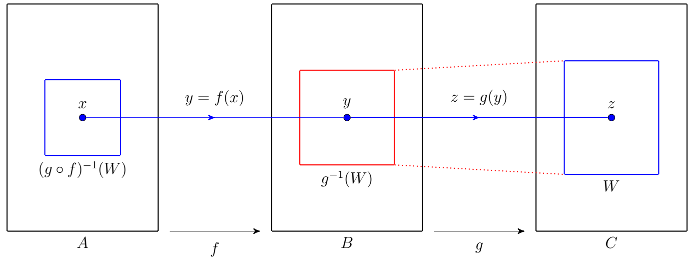
(2) Then there exists some $z \in W$ such that $z = (g \circ f)(x)$.



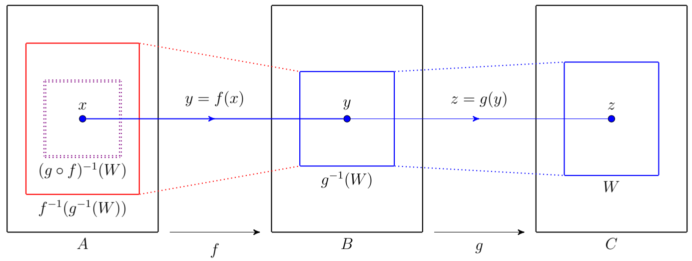
(3) We have $z = g(f(x))$.
Take $y = f(x)$. Note that $y \in B$.



(4) We have $g(y) = g(f(x)) = z$ and $z \in W$.
Then $y \in g^{-1}(W)$.



(5) We have $f(x) = y$ and $y \in g^{-1}(W)$.
Then $x \in f^{-1}(g^{-1}(W))$.



It follows that
$(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.

Proof of Statement (4b) of Theorem (4).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions. Let W be a subset of C.

[We want to prove $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.]

(α) ... It follows that $(g \circ f)^{-1}(W) \subset f^{-1}(g^{-1}(W))$.

(β) [We are going to prove that $(g \circ f)^{-1}(W) \supset f^{-1}(g^{-1}(W))$.]

[For any x, if $x \in f^{-1}(g^{-1}(W))$ then $x \in (g \circ f)^{-1}(W)$.]

Pick any object x.

Suppose $x \in f^{-1}(g^{-1}(W))$.

Then, by the definition of pre-image sets, there exists some $y \in g^{-1}(W)$ such that $y = f(x)$.

Now $y \in g^{-1}(W)$.

Then, by the definition of pre-image sets, there exists some $z \in W$ such that $z = g(y)$.

We have $z \in W$ and $z = g(y) = g(f(x)) = (g \circ f)(x)$. Then $x \in (g \circ f)^{-1}(W)$.

It follows that $f^{-1}(g^{-1}(W)) \subset (g \circ f)^{-1}(W)$.

It follows that $(g \circ f)^{-1}(W) = f^{-1}(g^{-1}(W))$.

12. **Theorem (5)**.

Let A, B be sets, and $f : A \longrightarrow B$ be a function. The following statements hold:

(5a) *Let S be a subset of A. $f^{-1}(f(S)) \supset S$.*

(5b) *Let U be a subset of B. $f(f^{-1}(U)) \subset U$.*

(5c) *Let S be a subset of A. $f(f^{-1}(f(S))) = f(S)$.*

(5d) *Let U be a subset of B. $f^{-1}(f(f^{-1}(U))) = f^{-1}(U)$.*

(5e) *Let S be a subset of A, and U be a subset of B. $f(S \cap f^{-1}(U)) = f(S) \cap U$.*

13. **Definition.**

Let A, B be sets and $f : A \longrightarrow B$ be a function. f is said to be **surjective** *if the statement (S) holds:*

(S): For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.

Theorem (6). (Characterizations of surjectivity).

Let A, B be sets and $f : A \longrightarrow B$ be a function.

The following statements are equivalent:

(I) *f is surjective.* (Ia) $f(A) = B$. (Ib) $f(A) \supset B$.

(II) *For any subset U of B, $f(f^{-1}(U)) \supset U$.*

(IIa) *For any subset U of B, $f(f^{-1}(U)) = U$.*

(III) *For any subset U of B, there exists some subset S of A such that $U = f(S)$.*

(IV) *For any subset T of A, $f(A \backslash T) \supset B \backslash f(T)$.*

(V) *For any subset U, V of B, if $f^{-1}(U) \subset f^{-1}(V)$ then $U \subset V$.*

(VI) *For any subset U, V of B, if $f^{-1}(U) = f^{-1}(V)$ then $U = V$.*

Proof of Theorem (6)?

14. **Definition.**

Let A, B be sets and $f : A \longrightarrow B$ be a function. f is said to be **injective** *if the statement (I) holds:*

(I): *For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.*

Theorem (7). (Characterizations of injectivity).

Let A, B be sets and $f : A \longrightarrow B$ be a function.

The following statements are equivalent:

(I) *f is injective.*

(II) *For any subset S of A, $f^{-1}(f(S)) \subset S$.*

(IIa) *For any subset S of A, $f^{-1}(f(S)) = S$.*

(III) *For any subset S of A, there exists some subset U of B such that $S = f^{-1}(U)$.*

(IV) *For any subset S, T of A, $f(S \cap T) \supset f(S) \cap f(T)$.*

(IVa) *For any subset S, T of A, $f(S \cap T) = f(S) \cap f(T)$.*

(V) *For any subsets S, T of A, if $f(S) \subset f(T)$ then $S \subset T$.*

(VI) *For any subsets S, T of A, if $f(S) = f(T)$ then $S = T$.*

15. **Theorem (8).**

Let A, B be sets and $f : A \longrightarrow B$ be a function.

(8a) *Let $\{U_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of B. ($\{f^{-1}(U_n)\}_{n=0}^{\infty}$ is an infinite sequence of subsets of A.) The following statements hold:*

(1) $f^{-1}(\bigcap\limits_{n=0}^{\infty} U_n) = \bigcap\limits_{n=0}^{\infty} f^{-1}(U_n)$.

(2) $f^{-1}(\bigcup\limits_{n=0}^{\infty} U_n) = \bigcup\limits_{n=0}^{\infty} f^{-1}(U_n)$.

(8b) *Let $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of A. ($\{f(S_n)\}_{n=0}^{\infty}$ is an infinite sequence of subsets of B.) The following statements hold:*

(1) $f(\bigcap\limits_{n=0}^{\infty} S_n) \subset \bigcap\limits_{n=0}^{\infty} f(S_n)$.

(2) $f(\bigcup\limits_{n=0}^{\infty} S_n) = \bigcup\limits_{n=0}^{\infty} f(S_n)$.

(8c) *The statements below are logically equivalent:*

(i) *f is injective.*

(ii) *For any infinite sequence of subsets $\{S_n\}_{n=0}^{\infty}$ of A, $f(\bigcap\limits_{n=0}^{\infty} S_n) = \bigcap\limits_{n=0}^{\infty} f(S_n)$.*