

1. In school maths we encountered the notion of locus for a ‘point particle’ ‘moving’ in a plane or in the space. For instance, consider the passage below:

- A ‘point particle’ is moving in the plane from time  $a$  to time  $b$ . At any time  $t$ , its position in the plane is given by the coordinates  $(u(t), v(t))$ . . . .

Translated into the the language of function, what this passage is about is the function  $f : [a, b] \rightarrow \mathbb{R}^2$  given  $f(t) = (u(t), v(t))$  for any  $t \in [a, b]$ . The position of the ‘point particle’ ‘at time  $t$ ’ is the value of  $f(t)$ . The locus of the ‘point particle’ is the image set of  $[a, b]$  under  $f$ .

So we did actually come across the notion of image set under a function, without knowing its name.

2. **Parametrized curves.**

Suppose we are given a ‘nice’ curve  $C$  in  $\mathbb{R}^n$ . To ‘describe’ it we may think of the curve  $C$  (except perhaps a few points) as the image set of some subset  $I$  of  $\mathbb{R}$ , under some function  $f$  of one real variable. Very often  $I$  is so nice that it is an interval.  $f$  is so nice that it is differentiable, (or even better, smooth), and when it is restricted to  $I$ , it is injective.

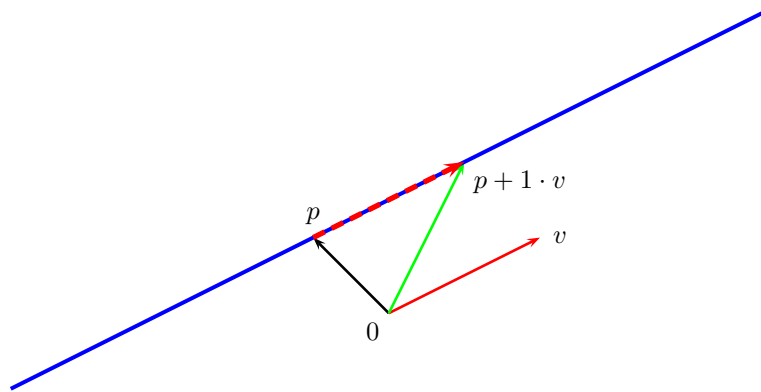
In this sense we say we are ‘parametrizing’ the curve  $C$ ; the ‘parametrization’ under question is the function  $f$ . Equivalently  $C$  is the ‘trace’ of the ‘parametrized curve’  $f$ . When the points in the set  $I$  is interpreted as time, the curve  $C$  is the locus of some ‘point particle’ moving in the plane/space according to some rule specified by the function  $f$ .

3. **Examples of parametrized curves.**

- (a) Let  $p \in \mathbb{R}^n$ ,  $v \in \mathbb{R}^n \setminus \{0\}$ . (Think of  $p, v$  as vectors.)

Define the function  $f : \mathbb{R} \rightarrow \mathbb{R}^n$  by  $f(t) = p + tv$  for any  $t \in \mathbb{R}$ .

$f(\mathbb{R})$  is the ‘infinite’ straight line in  $\mathbb{R}^n$  passing through  $p$  and being parallel to  $v$ .

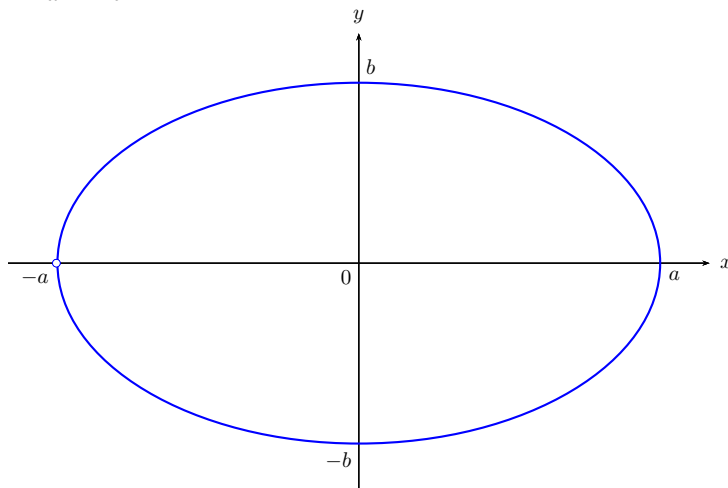


Suppose  $a, b \in \mathbb{R}$ , and  $a < b$ .  $f([a, b])$  is the line segment joining the points  $p + av$ ,  $p + bv$ .

- (b) Let  $a, b \in (0, +\infty)$ .

Define the function  $g : (-\pi, \pi) \rightarrow \mathbb{R}^2$  by  $g(t) = (a \cos(t), b \sin(t))$  for any  $t \in (-\pi, \pi)$ .

$g$  ‘parametrizes’ the **ellipse**  $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$  with the point  $(-a, 0)$  removed.

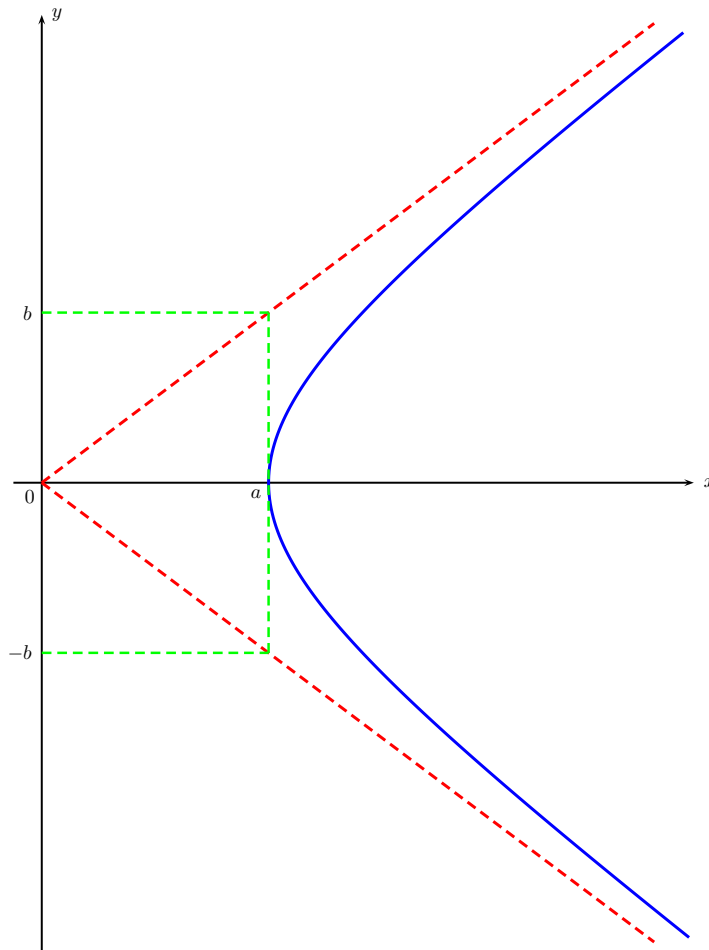


Suppose  $\sigma, \tau \in (-\pi, \pi)$ , and  $\sigma < \tau$ .  $g([\sigma, \tau])$  is the ‘elliptic arc’ of ‘between  $t = \sigma$  and  $t = \tau$ ’.

(c) Let  $a, b \in (0, +\infty)$ .

Define the function  $g : \mathbb{R} \rightarrow \mathbb{R}^2$  by  $g(t) = (a \cosh(t), b \sinh(t))$  for any  $t \in \mathbb{R}$ .

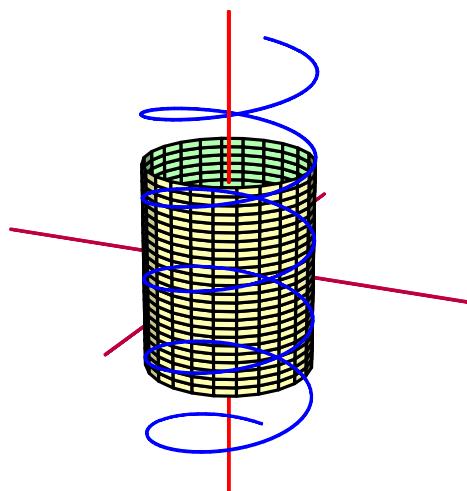
$g$  ‘parametrizes’ one branch of the **hyperbola**  $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ .



Suppose  $\sigma, \tau \in (-\pi, \pi]$ , and  $\sigma < \tau$ .  $g([\sigma, \tau])$  is the ‘hyperbolic arc’ of ‘between  $t = \sigma$  and  $t = \tau$ ’.

(d) Define the function  $h : \mathbb{R} \rightarrow \mathbb{R}^3$  by  $h(t) = (\cos(2\pi t), \sin(2\pi t), t)$  for any  $t \in \mathbb{R}$ .

$h$  ‘parametrizes’ the **circular helix** passing through the point  $(1, 0, 0)$  with gauge 1 on the cylinder  $x^2 + y^2 = 1$ .



Suppose  $a, b \in \mathbb{R}$  and  $a < b$ .  $g([a, b])$  is the ‘section’ of this space curve ‘between  $t = a$  and  $t = b$ ’.

#### 4. Parametrized surfaces.

The idea in the notion of parametrized curves can be generalized to give the notion of parametrized surfaces (and beyond).

Suppose we are given a ‘nice’ surface  $S$  in  $\mathbb{R}^n$ . To ‘describe’ it we may think of the surface  $S$  (except perhaps a few points or a few curves on it) as the image set of some subset  $D$  of  $\mathbb{R}^2$  under some function  $f$  of two real variables.

Very often  $D$  is so nice that it is a plane figure that we know well, for instance, a rectangular region, or a triangular region, or a disc.  $f$  is so nice that it is differentiable everywhere (or even better, smooth), and when it is restricted to  $D$ , it is injective.

In this sense we say we are ‘parametrizing’ the surface  $S$ ; the ‘parametrization’ under question is the function  $f$ . Equivalently  $S$  is the ‘trace’ of the ‘parametrized surface’  $f$ .

The restriction of  $f$  to any specific constant value, say,  $x_0$ , of its ‘first real variable’, defines the parametrized curve whose ‘formula of definition’ is given by  $y \mapsto f(x_0, y)$  whenever  $(x_0, y) \in D$ .

The restriction of  $f$  to any specific constant value, say,  $y_0$ , of its ‘second real variable’, defines the parametrized curve whose ‘formula of definition’ is given by  $x \mapsto f(x, y_0)$  whenever  $(x, y_0) \in D$ .

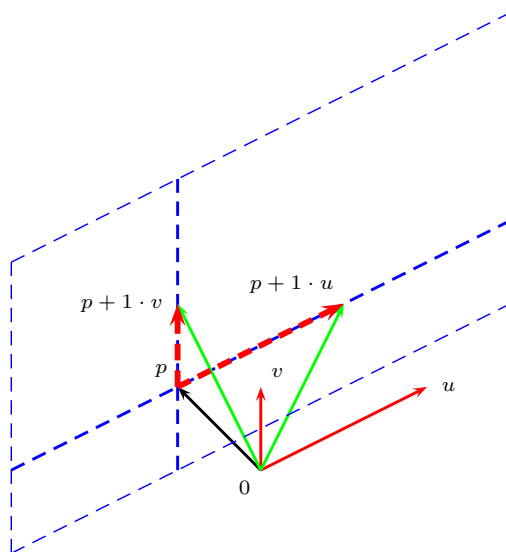
In this way we can visualize the surface  $S$  as the unions of two ‘families of parametrized curves’, obtained from restricting  $f$  to various specific values of its ‘first real variable’ and its ‘second real variable’ respectively.

### 5. Examples of parametrized surfaces.

(a) Let  $p \in \mathbb{R}^n$ ,  $u, v \in \mathbb{R}^n \setminus \{0\}$ . Suppose  $u, v$  are linearly independent.

Define the function  $f : \mathbb{R}^2 \rightarrow \mathbb{R}^n$  by  $f(s, t) = p + su + tv$  for any  $s, t \in \mathbb{R}$ .

$f(\mathbb{R}^2)$  is the ‘infinite’ plane in  $\mathbb{R}^n$  passing through  $p$  and being parallel to ‘span’ of  $u, v$ . The vector  $u \times v$  is perpendicular to this plane.



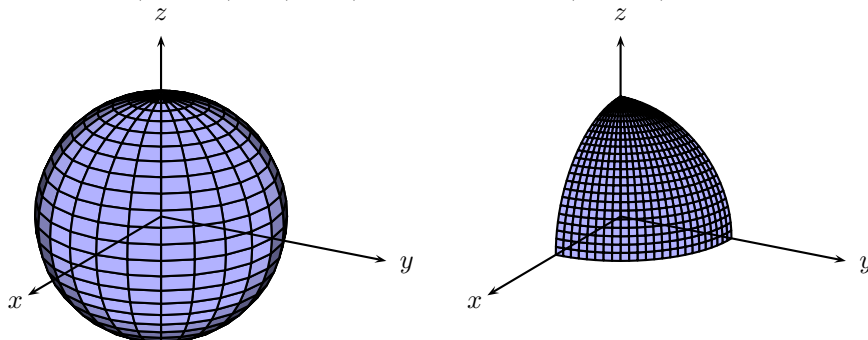
(b) Define the function  $g : (-\pi, \pi) \times \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \rightarrow \mathbb{R}^3$  by

$$g(\theta, \varphi) = (\cos(\theta) \cos(\varphi), \sin(\theta) \cos(\varphi), \sin(\varphi))$$

for any  $\theta \in (-\pi, \pi)$ ,  $\varphi \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ .

$g$  ‘parametrizes’ the ‘slitted’ unit sphere  $x^2 + y^2 + z^2 = 1$  with centre at origin.

The ‘slit’ is the circular arc from  $(0, 0, -1)$  to  $(0, 0, 1)$  passing through  $(-1, 0, 0)$ .



$g([0, \pi/2] \times [0, \pi/2))$  is one quarter of the ‘northern’ hemisphere.

(c) Let  $r > 0$  and  $R > 0$ . Suppose  $R > r$ .

Define the function  $h : (-\pi, \pi) \times (-\pi, \pi) \rightarrow \mathbb{R}^3$  by

$$h(\theta, \varphi) = ((R + r \cos(\varphi)) \cos(\theta), (R + r \cos(\varphi)) \sin(\theta), r \sin(\varphi))$$

for any  $\theta \in (-\pi, \pi)$ ,  $\varphi \in (-\pi, \pi)$ .

$h$  'parametrizes' the 'slitted' torus  $(\sqrt{x^2 + y^2} - R)^2 + z^2 = r^2$  with centre at origin, 'floated' on the  $xy$ -plane, obtained by rotating about the  $z$ -axis the circle on the  $xz$ -plane with centre at  $(R, 0, 0)$  and with radius  $r$ .

The 'slits' are the union of two circles: the circle on the  $xz$ -plane with centre at  $(-R, 0, 0)$  and with radius  $r$ , and the circle on the  $xy$ -plane with centre at  $(0, 0, 0)$  and with radius  $R - r$ .

