

1. **Definition.**

Let S be a subset of \mathbb{R} . The set S is said to be an **interval in \mathbb{R}** if any one of the statements below hold:

- | | |
|---|---|
| (a) $S = \emptyset$. | (f) $S = \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R}$. |
| (b) $S = \{x \in \mathbb{R} : a < x\}$ for some $a \in \mathbb{R}$. | (g) $S = \{x \in \mathbb{R} : a \leq x < b\}$ for some $a, b \in \mathbb{R}$. |
| (c) $S = \{x \in \mathbb{R} : a \leq x\}$ for some $a \in \mathbb{R}$. | (h) $S = \{x \in \mathbb{R} : a < x \leq b\}$ for some $a, b \in \mathbb{R}$. |
| (d) $S = \{x \in \mathbb{R} : x < b\}$ for some $b \in \mathbb{R}$. | (i) $S = \{x \in \mathbb{R} : a \leq x \leq b\}$ for some $a, b \in \mathbb{R}$. |
| (e) $S = \{x \in \mathbb{R} : x \leq b\}$ for some $b \in \mathbb{R}$. | (j) $S = \mathbb{R}$. |

Remark on notations and terminologies. We write:

- | | |
|--|---|
| • $(a, +\infty) = \{x \in \mathbb{R} : a < x\}$ | • $(a, b) = \{x \in \mathbb{R} : a < x < b\}$ |
| • $[a, +\infty) = \{x \in \mathbb{R} : a \leq x\}$ | • $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$ |
| • $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$ | • $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$ |
| • $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$ | • $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$ |

Each of the numbers a, b is called an endpoint of the interval concerned. If the interval concerned contains all its endpoints as its elements, it is said to be a **closed interval**. Where it contains none, it is said to be an **open interval**. If the interval is bounded in \mathbb{R} , it is said to be a **bounded interval**. If it is not bounded in \mathbb{R} , it is said to be an **unbounded interval**.

2. **Theorem (1). (Characterization of intervals.)**

Let S be a subset of \mathbb{R} . The statements (I1), (I2) below are equivalent:

- (I1) S is an interval.
 (I2) For any $x, y \in S$, for any $u \in \mathbb{R}$, if $x \leq u \leq y$ then $u \in S$.

3. **Outline of a proof of Theorem (1).**

The argument for ‘(I1) \implies (I2)’ is tedious but straightforward. Below is an outline of the argument for ‘(I2) \implies (I1)’:

- (a) Let S be a subset of \mathbb{R} . Suppose S satisfies (I2).
 If $S = \emptyset$ then S is an interval.
 From now on, suppose $S \neq \emptyset$.
- (b) Suppose S is neither bounded above in \mathbb{R} nor bounded below in \mathbb{R} . Then, by applying (I2), we deduce $\mathbb{R} \subset S$. It follows that $S = \mathbb{R}$.
- (c) Suppose S is bounded above in \mathbb{R} , and S is not bounded below in \mathbb{R} .
 i. Take some $c \in S$. By applying (I2), we deduce that $(-\infty, c] \subset S$.
 ii. Since S is non-empty and is bounded above in \mathbb{R} , S has a supremum in \mathbb{R} , which we denote by b .
 iii. Since b is an upper bound of S in \mathbb{R} , we have $S \subset (-\infty, b]$.
 iv. By applying (I2), we deduce that $(c, b) \subset S$. (Why? Pick any $u \in (c, b)$. Since $k < b$, u is not an upper bound of S in \mathbb{R} . Then there exists some $v \in S$ such that $v > u$. Since $v \in S$, we have $v \leq b$. Then $c < k < v$. By (I2), $k \in S$.)
 v. It follows that $(-\infty, b) \subset S \subset (-\infty, b]$. Then $S = (-\infty, b)$ or $(-\infty, b]$.
- (d) Suppose S is bounded below in \mathbb{R} , and S is not bounded above in \mathbb{R} .
 Modifying the argument above, we deduce that S has an infimum in \mathbb{R} , which we denote by a , and that furthermore, $(a, +\infty) \subset S \subset [a, +\infty)$. Then $S = (a, +\infty)$ or $[a, +\infty)$.
- (e) Suppose S is bounded below in \mathbb{R} and bounded above in \mathbb{R} .
 Modifying the argument above, we deduce that S has an infimum and a supremum in \mathbb{R} , we denote by a, b respectively, and that furthermore, $(a, b) \subset S \subset [a, b]$.
 Then $S = (a, b)$ or $S = (a, b]$ or $S = [a, b)$ or $S = [a, b]$.

4. **Theorem (2).**

Let J be an interval in \mathbb{R} , and $g : J \rightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J . Then $g(J)$ is an interval.

Remark. The proof of Theorem (2) relies on Theorem (1) and the Intermediate Value Theorem.

5. Intermediate Value Theorem.

Let $a, b \in \mathbb{R}$, with $a < b$. Let $h : [a, b] \rightarrow \mathbb{R}$ be a function. Suppose $h(a) \neq h(b)$. Suppose h is continuous on $[a, b]$. Then, for any $\gamma \in \mathbb{R}$, if γ is strictly between $h(a)$ and $h(b)$ then there exists some $c \in (a, b)$ such that $h(c) = \gamma$.

6. Proof of Theorem (2).

Let J be an interval in \mathbb{R} , and $g : J \rightarrow \mathbb{R}$ be a function.

Suppose g is continuous on J .

[We want to verify that for any $s, t \in g(J)$, for any $u \in \mathbb{R}$, if $s \leq u \leq t$ then $u \in g(J)$.]

Pick any $s, t \in g(J)$. Pick any $u \in \mathbb{R}$. Suppose $s \leq u \leq t$.

If $s = t$ or $u = s$ or $u = t$, then $u \in g(J)$ trivially. From now on suppose $s < u < t$.

By the definition of $g(J)$, there exists some $a, b \in J$ such that $s = g(a)$ and $t = g(b)$. Since $s < t$, we have $a \neq b$.

By assumption g is continuous on J . Since J is an interval and $a, b \in J$, the closed and bounded interval I with endpoints a, b lies entirely in J . Then g is continuous on I .

By the Intermediate Value Theorem, there exists some c strictly between a, b such that $u = g(c)$. Then $u \in g(J)$.

Now by Theorem (1), $g(J)$ is an interval.

7. Theorem (3).

Let K be a closed and bounded interval in \mathbb{R} , and $g : K \rightarrow \mathbb{R}$ be a function.

Suppose g is continuous on K . Then $g(K)$ is a closed and bounded interval. Moreover, the endpoints of $g(K)$ are respectively the least element and the greatest element of $g(K)$.

Remark. The proof of Theorem (3) relies on Theorem (2), the Intermediate Value Theorem and the Existence-of-Extremum Theorem for continuous functions.

Further remark. The statements below are false:

- (a) Let J be an open interval in \mathbb{R} , and $g : J \rightarrow \mathbb{R}$ be a function.
Suppose g is continuous on J . Then $g(J)$ is an open interval.
- (b) Let J be a closed interval in \mathbb{R} , and $g : J \rightarrow \mathbb{R}$ be a function.
Suppose g is continuous on J . Then $g(J)$ is a closed interval.
- (c) Let J be a bounded interval in \mathbb{R} , and $g : J \rightarrow \mathbb{R}$ be a function.
Suppose g is continuous on J . Then $g(J)$ is a bounded interval.
- (d) Let J be an unbounded interval in \mathbb{R} , and $g : J \rightarrow \mathbb{R}$ be a function.
Suppose g is continuous on J . Then $g(J)$ is an unbounded interval.

8. Definition. (Absolute extrema for real-valued functions of one real variable.)

Let I be an interval, and $h : D \rightarrow \mathbb{R}$ be a real-valued function of one real variable with domain D which contains I as a subset entirely. Let p be a point in I .

- (a) h is said to **attain absolute maximum at p on I** if for any $x \in I$, the inequality $h(x) \leq h(p)$ holds. The number $h(p)$ is called the **absolute maximum value of h on I** .
- (b) h is said to **attain absolute minimum at p on I** if for any $x \in I$, the inequality $h(x) \geq h(p)$ holds. The number $h(p)$ is called the **absolute minimum value of h on I** .
- (c) h is said to **attain (absolute) extremum at p on I** if h attains absolute maximum at p or h attains absolute minimum at p .

Existence-of-Extremum Theorem for continuous functions.

Let I be a closed and bounded interval in \mathbb{R} , and $f : I \rightarrow \mathbb{R}$ be a function.

Suppose f is continuous on I . Then there exist some $p, q \in I$ such that f attains absolute minimum at p on I and f attains absolute maximum at q on I .

Remark. The key step in the proof of the Existence-of-Extremum Theorem for continuous functions is to prove that the function f is bounded, in the sense that there exist some positive real number C such that for any $x \in I$, $|f(x)| \leq C$. The technical detail for this step is beyond the scope of this course: it relies on ideas that will be introduced in your *analysis* course.

9. Proof of Theorem (3).

Let K be a closed and bounded interval in \mathbb{R} , and $g : K \rightarrow \mathbb{R}$ be a function.

Suppose g is continuous on K .

By Theorem (2), $g(K)$ is an interval.

By the Existence-of-Extremum Theorem for continuous functions, there exist some $p, q \in K$ such that g attains absolute minimum at p on K and g attains absolute maximum at q on K .

We verify that $g(K) = [g(p), g(q)]$:

- Pick any $y \in [g(p), g(q)]$. Then $g(p) \leq y \leq g(q)$. If $y = g(p)$ or $y = g(q)$ then $y \in g(K)$.
From now on suppose $g(p) < y < g(q)$. Then by the Intermediate Value Theorem, there exists some x strictly between p and q such that $y = g(x)$.
Since K is an interval, $p, q \in K$ and x is strictly between p, q , we have $x \in K$.
Then $y \in g(K)$.
- Suppose $v \in g(K)$. Then there exists some $u \in K$ such that $v = g(u)$.
Since g attains absolute minimum at p on K , we have $v = g(u) \geq g(p)$.
Since g attains absolute maximum at q on K , we have $v = g(u) \leq g(q)$.
Then $g(p) \leq v \leq g(q)$. Therefore $v \in [g(p), g(q)]$.

Hence $g(K)$ is the closed and bounded interval $[g(p), g(q)]$, whose least element and greatest element are respectively $g(p), g(q)$.

10. Definition.

Let S be a subset of \mathbb{R} .

- (a) S is said to be **open in \mathbb{R}** if for any $x \in S$, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \subset S$.
- (b) S is said to be **closed in \mathbb{R}** if $\mathbb{R} \setminus S$ is open in \mathbb{R} .

Remark.

- (a) \emptyset is open in \mathbb{R} and is closed in \mathbb{R} .
- (b) \mathbb{R} is open in \mathbb{R} and is closed in \mathbb{R} .
- (c) Every open interval in \mathbb{R} is open in \mathbb{R} .
- (d) Every closed interval in \mathbb{R} is closed in \mathbb{R} .

11. Definition.

Let A be a subset of \mathbb{R} , and S be a subset of A .

- (a) S is said to be **open in A** if for any $x \in S$, there exists some $\delta > 0$ such that $(x - \delta, x + \delta) \cap A \subset S$.
- (b) S is said to be **closed in A** if $A \setminus S$ is open in A .

Lemma (4).

Let A be a subset of \mathbb{R} , and S be a subset of A . The statements below are logically equivalent:

- (a) S is open in A .
- (b) There exists some subset T of \mathbb{R} such that T is open in \mathbb{R} and $S = T \cap A$.

Remark. The proof of Lemma (4) is easy.

12. Theorem (5).

Let D be a subset of \mathbb{R} , and $f : D \rightarrow \mathbb{R}$ be a function.

The statements below are logically equivalent:

- (a) f is continuous on D .
- (b) For any subset U of \mathbb{R} , if U is open in \mathbb{R} then $f^{-1}(U)$ is open in D .
- (c) For any subset J of \mathbb{R} , if J is an open interval in \mathbb{R} then $f^{-1}(J)$ is open in D .

Remark. Theorem (5) is a straightforward consequence of the definition of pre-image set, the definition of open set in \mathbb{R} , and the (formal) definition for the notion of continuity.

Definition.

Let A be a subset of \mathbb{R} , and $h : A \rightarrow \mathbb{R}$ be a function. Let $c \in A$.

h is said to be **continuous at c** if the statement (CT) holds:

(CT) For any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $x \in A$, if $|x - c| < \delta$ then $|h(x) - h(c)| < \varepsilon$.

Furthermore, h is said to be **continuous on D** if h is continuous at every point of D .

13. **Proof of Theorem (5).**

Let D be a subset of \mathbb{R} , and $f : D \rightarrow \mathbb{R}$ be a function.

- [(a) \implies (b)?]

Suppose f is continuous on D .

[We want to prove that for any subset U of \mathbb{R} , if U is open in \mathbb{R} then $f^{-1}(U)$ is open in D .]

Let U be a subset of \mathbb{R} . Suppose U is open in \mathbb{R} .

[We want to verify that $f^{-1}(U)$ is open in \mathbb{R} .]

Pick any $c \in f^{-1}(U)$. By the definition of pre-image set, we have $f(c) \in U$.

Since U is open in \mathbb{R} , there exists some $\eta > 0$ such that $(f(c) - \eta, f(c) + \eta) \subset U$.

By continuity, for the same $\eta > 0$, there exists some $\delta > 0$ such that for any $x \in D$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \eta$.

We verify that $(c - \delta, c + \delta) \cap D \subset f^{-1}(U)$:

Pick any $x \in (c - \delta, c + \delta) \cap D$. We have $x \in D$ and $|x - c| < \delta$. Then (by continuity,) $|f(x) - f(c)| < \eta$. Therefore $f(x) \in (f(c) - \eta, f(c) + \eta)$. Hence $f(x) \in U$.

By the definition of pre-image set, we have $x \in f^{-1}(U)$.

It follows that $f^{-1}(U)$ is open in \mathbb{R} .

- [(b) \implies (c)?]

Suppose that for any subset U of \mathbb{R} , if U is open in \mathbb{R} then $f^{-1}(U)$ is open in D .

[We want to prove that for any subset J of \mathbb{R} , if J is an open interval in \mathbb{R} then $f^{-1}(J)$ is open in D .]

Let J be a subset of \mathbb{R} . Suppose J is an open interval in \mathbb{R} .

Note that J is open in \mathbb{R} . Then, by assumption, $f^{-1}(J)$ is open in D .

- [(c) \implies (a)?]

Suppose that for any subset J of \mathbb{R} , if J is an open interval in \mathbb{R} then $f^{-1}(J)$ is open in D .

[We want to prove that f is continuous on D . This amounts to verify that for any $c \in D$, for any $\varepsilon > 0$, there exists some $\delta > 0$ such that for any $x \in D$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$.]

Pick any $c \in D$. Pick any $\varepsilon > 0$. Write $J = (f(c) - \varepsilon, f(c) + \varepsilon)$. Note that J is an open interval in \mathbb{R} .

By assumption, $f^{-1}(J)$ is open in D .

Since $f(c) \in J$, we have $c \in J$ by the definition of pre-image set.

Then, by the definition of open set, there exists some $\delta > 0$ such that $(c - \delta, c + \delta) \cap D \subset f^{-1}(J)$.

[We now ask: Is it true that for any $x \in D$, if $|x - c| < \delta$ then $|f(x) - f(c)| < \varepsilon$?]

Pick any $x \in D$. Suppose $|x - c| < \delta$. Then $x \in (c - \delta, c + \delta)$ and $x \in D$. Therefore $x \in (c - \delta, c + \delta) \cap D$. Hence $x \in f^{-1}(J)$.

Now by the definition of pre-image set, we have $f(x) \in J$. Then $|f(x) - f(c)| < \varepsilon$.

It follows that f is continuous at c .