1. **Definition**.

Let A, B be sets and $f : A \longrightarrow B$ be a function from A to B.

Let S be a subset of A.

The **image set of the set S under the function f** is defined to be the set

$$\{y \in B : \text{There exists some } x \in S \text{ such that } y = f(x)\}$$

(or $\{y \in B : y = f(x) \text{ for some } x \in S\}$, or $\{f(x) \mid x \in S\}$).

It is denoted by $f(S)$.

[handwritten annotations: $Q(x,y)$ is '$y = f(x)$'. $(\exists x \in S) \, Q(x,y)$. This is a predicate with variable y.]

Remarks.

(1) **Terminology.** We write $f(A) = \{y \in B : y = f(x) \text{ for some } x \in A\}$.

The set $f(A)$ is called the **image of the function f**.

(2) **Warning.** '$f(S)$' should be understood as one symbol which is the name of some subset of B whose 'content' depends on f and S, according to the predicate

'there exists some $x \in S$ such that $y = f(x)$'

for its construction via the Method of Specification.

2. Example of pictorial visualizations of image sets.

Let $A = \{k, \ell, m, n, p, q, r, \cdots\}$, $B = \{c, d, e, g, h, i, j, \cdots\}$, and $f : A \longrightarrow B$ be the function defined by $f(k) = d$, $f(\ell) = f(m) = f(n) = e$, $f(p) = g$, $f(q) = f(r) = h$, ... Let $S = \{m, n, p, q\}$. We have $f(S) = \{f(m), f(n), f(p), f(q)\} = \{e, g, h\}$.
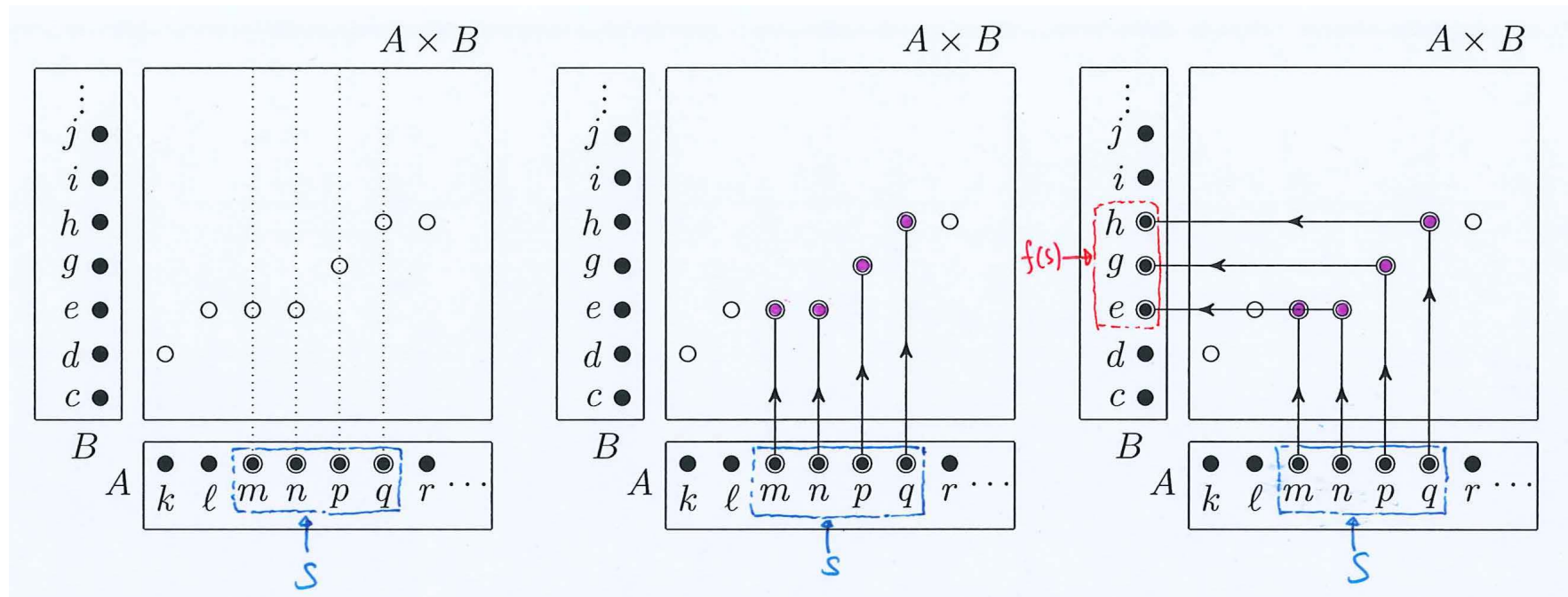
(a) 'Blobs-and-arrows diagram'.

Two equivalent ways to see what $f(S)$ is:

(i) Examine each $y \in B$. Ask if it is targetted by some $x \in S$ via f. If yes, collect it; if no, throw away.

(ii) Simply collect the target of every $x \in S$.
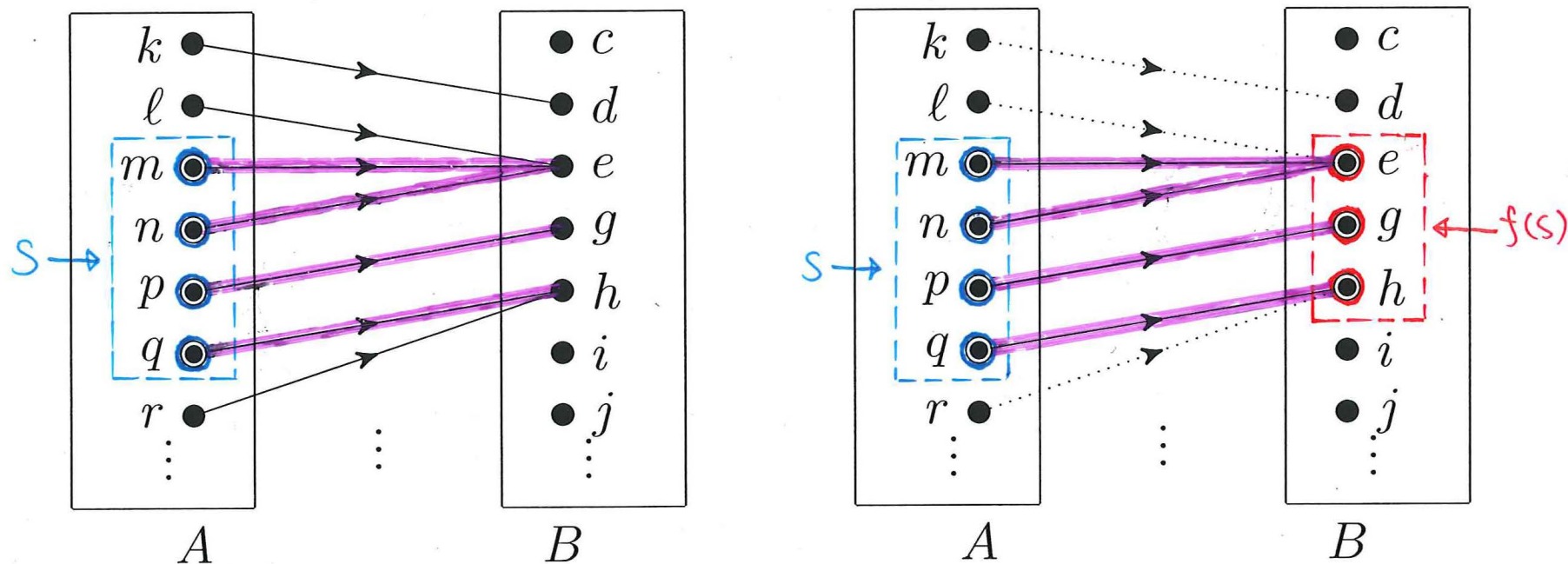
(b) 'Coordinate plane diagram'.

3. **Definition**.

Let A, B be sets and $f : A \longrightarrow B$ be a function from A to B.

Let U be a subset of B.

The **pre-image set of the set U under the function f** is defined to be the set

$$\{x \in A : \text{There exists some } y \in U \text{ such that } y = f(x)\}$$

[handwritten annotations: $Q(x,y)$ is '$y = f(x)$'. $\{(\exists y \in U)\, Q(x,y)\}$ This is a predicate with variable x.]

(or $\{x \in A : y = f(x) \text{ for some } y \in U\}$, or $\{x \in A : f(x) \in U\}$).

It is denoted by $f^{-1}(U)$.

Warnings.

(1) '$f^{-1}(U)$' should be understood as one symbol which is the name of some subset of A whose 'content' depends on f and U, according to the predicate

$$\text{'there exists some } y \in U \text{ such that } y = f(x)\text{'}$$

for its construction via the Method of Specification.

(2) The presence of the chain of symbols 'f^{-1}' in '$f^{-1}(U)$' does not give any hint as to whether f is 'invertible'/bijective as a function or not.

4. **Example of pictorial visualizations of pre-image sets.**

Let $A = \{k, \ell, m, n, p, q, r, \cdots\}$, $B = \{c, d, e, g, h, i, j, \cdots\}$, and $f : A \longrightarrow B$ be the function defined by $f(k) = d$, $f(\ell) = f(m) = f(n) = e$, $f(p) = g$, $f(q) = f(r) = h$, $\ldots$. Let $U = \{e, g, i\}$. We have $f^{-1}(U) = \{\ell, m, n, p\} = \{\ell, m, n\} \cup \{p\} \cup \emptyset = f^{-1}(\{e\}) \cup f^{-1}(\{g\}) \cup f^{-1}(\{i\})$.

(a) 'Blobs-and-arrows diagram'.

Two equivalent ways to see what $f^{-1}(U)$ is:

(i) Examine each $x \in A$. Ask if it targets some $y \in U$ via f. If yes, collect it; if no, throw away.

(ii) For each $y \in U$, collect whatever in A that targets it, and then 'take union'.
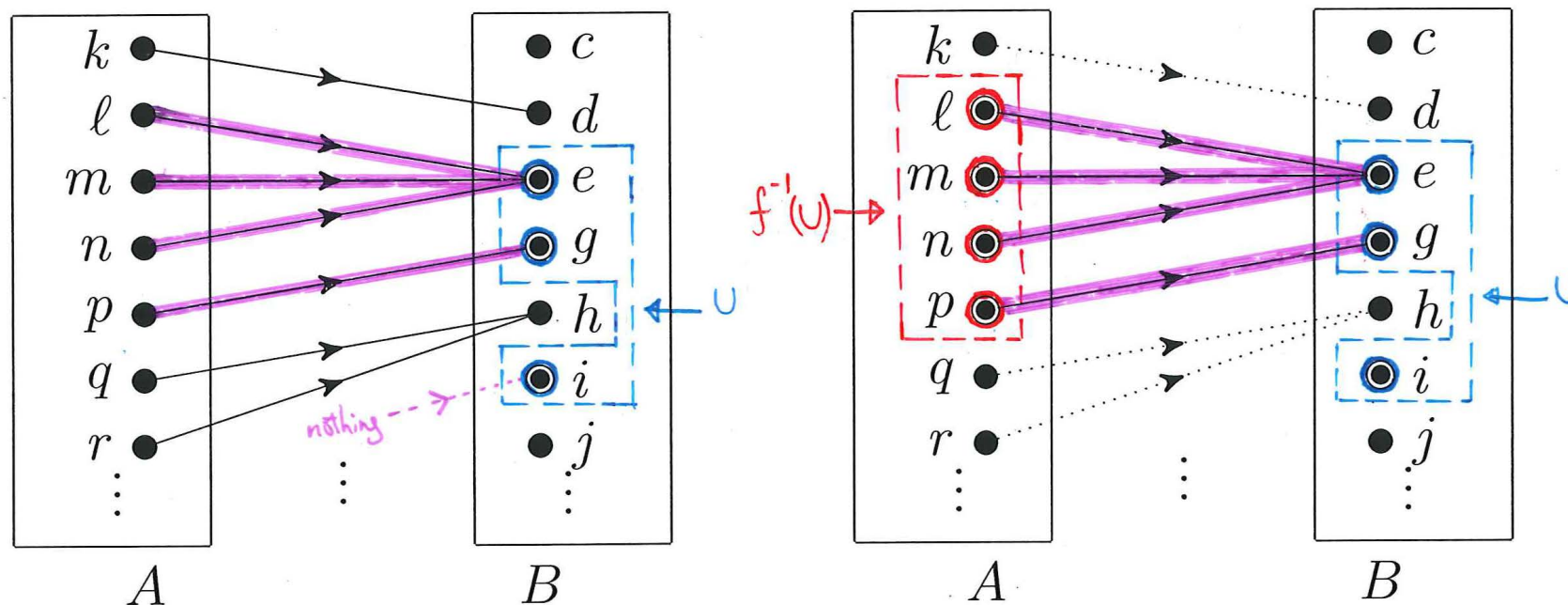
(b) 'Coordinate plane diagram'.

For each $y_0 \in B$, to determine the pre-image set $f^{-1}(\{y_0\})$ of the singleton $\{y_0\}$ under f is the same as:

- to determine the **solution set** of the equation $y_0 = f(x)$ with unknown x in A.

We call this set the **level set** at y_0 of the function f.

So $f^{-1}(U)$ is the 'union' of the level sets of the y's in U of the function f.
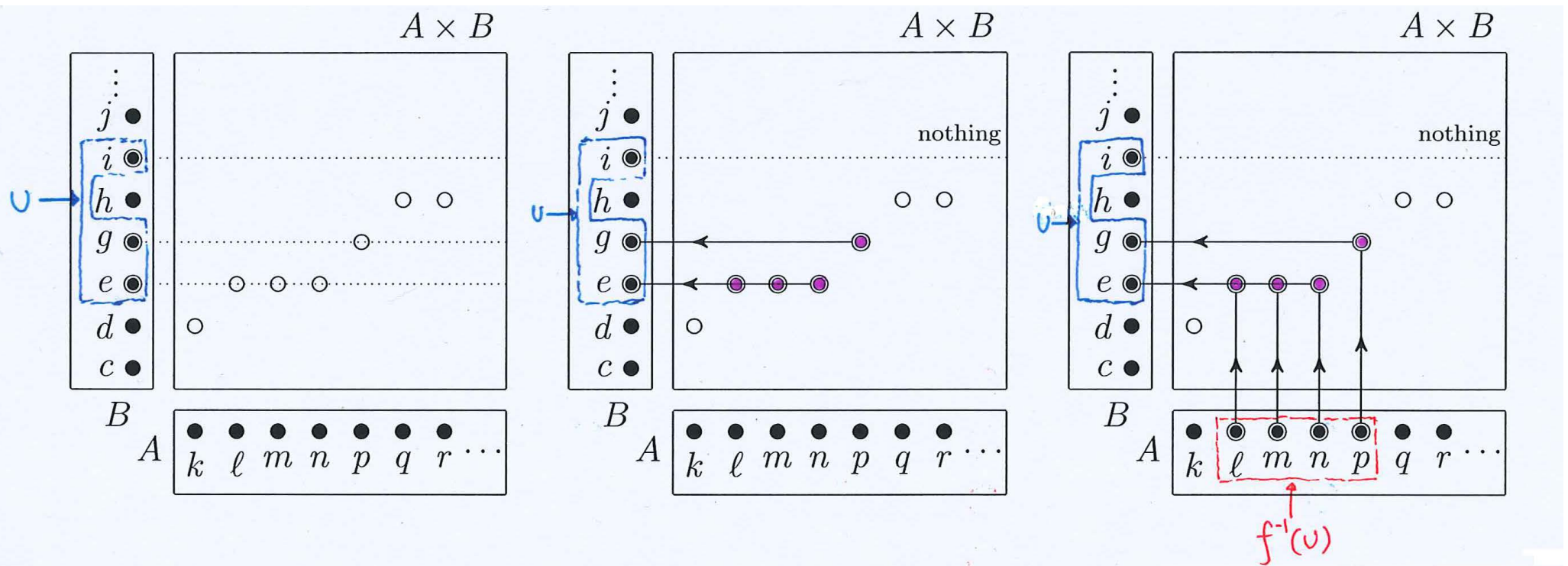
Solution set of the equation $e = f(u)$ with unknown u in A? $\{\ell, m, n\}$

Solution set of the equation $g = f(u)$ with unknown u in A? $\{p\}$

Solution set of the equation $i = f(u)$ with unknown u in A? $\emptyset$