1. **Formal definition for the notion of relations and functions, and re-formulation of the definition of surjectivity and injectivity.**

(a) *Let D, R, H be sets.*
The ordered triple (D, R, H) is called a **relation from D to R** *if $H \subset D \times R$.*

(b) *Let D, R be sets, and H be a subset of $D \times R$.*
The relation (D, R, H) is said to be a **function from domain D to range R with graph H** *if both of the statements $(E), (U)$ below hold:*

(E): *For any $t \in D$, there exists some $u \in R$ such that $(t, u) \in H$.*

(U): *For any $t \in D$, for any $u, v \in R$, if $(t, u) \in H$ and $(t, v) \in H$ then $u = v$.*

Where we refer to (D, R, H) as h, we write $u = h(t)$ exactly when $(t, u) \in H$.

(c) *Let D, R be sets, and $h : D \longrightarrow R$ be a function from D to R with graph H.*

i. *h is said to be* **surjective** *if the statement (S) below holds:*

(S): *For any $u \in R$, there exists some $t \in D$ such that $(t, u) \in H$.*

ii. *h is said to be* **injective** *if the statement (I) below holds:*

(I): *For any $u \in R$, for any $t, s \in D$, if $(t, u) \in H$ and $(s, u) \in H$ then $t = s$.*

2. **Definition for the notion of bijective function.**

Let D, R be sets, and $h : D \longrightarrow R$ be a function from D to R.

h is said to be **bijective** *if h is both surjective and injective.*

Remark.

Hence

$$h = (D, R, H) \text{ is a bijective function from } D \text{ to } R \text{ with graph } H$$
$$\text{iff}$$
$$\text{all of the statements } (E), (U), (S), (I) \text{ below hold:}$$

(E): *For any $t \in D$, there exists some $u \in R$ such that $(t, u) \in H$.*

(U): *For any $t \in D$, for any $u, v \in R$, if $(t, u) \in H$ and $(t, v) \in H$ then $u = v$.*

(S): *For any $u \in R$, there exists some $t \in D$ such that $(t, u) \in H$.*

(I): *For any $u \in R$, for any $t, s \in D$, if $(t, u) \in H$ and $(s, u) \in H$ then $t = s$.*

3. **Definition for the notion of inverse function.**

Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions.

g is said to be an **inverse function** *of f if both of the following statements hold:*

(†) *For any $x \in A$, $(g \circ f)(x) = x$.*

(‡) *For any $y \in B$, $(f \circ g)(y) = y$.*

Definition for the notion of identity function.

Let C be a set.

Define the function $\mathsf{id}_C : C \longrightarrow C$ by $\mathsf{id}_C(z) = z$ for any $z \in C$.

id_C is called the **identity function** *on the set C.*

Theorem (1). (Re-formulation of the definition of inverse function.)

Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions.

The statements below are logically equivalent:

($\star_0$) *g is an inverse function of f.*

($\star_1$) *$g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$ as functions.*

($\star_2$) *f is an inverse function of g.*

($\star_3$) *For any $x \in A$, for any $y \in B$, ($y = f(x)$ iff $x = g(y)$).*

4. **Theorem (2). (Uniqueness of inverse function.)**
Let A, B be sets, and $f : A \longrightarrow B$ be a function.
f has at most one inverse function.

Theorem (3). (Necessary condition for existence of inverse function.)
Let A, B be sets, $f : A \longrightarrow B$ be a function.
Suppose f has an inverse function, say, $g : B \longrightarrow A$.
Then each of f, g is bijective.

Question. *Is the necessary condition sufficient as well? Why?*
Answer. *Yes. Reason: Theorem (4).*

5. **Theorem (4). (Existence and Uniqueness of inverse function for a bijective function.)**

Let A, B be sets, and $f : A \longrightarrow B$ be a function.

Suppose f is bijective.

Then there exists some unique bijective function $g : B \longrightarrow A$ such that g is the inverse function of f.

Convention on notations.

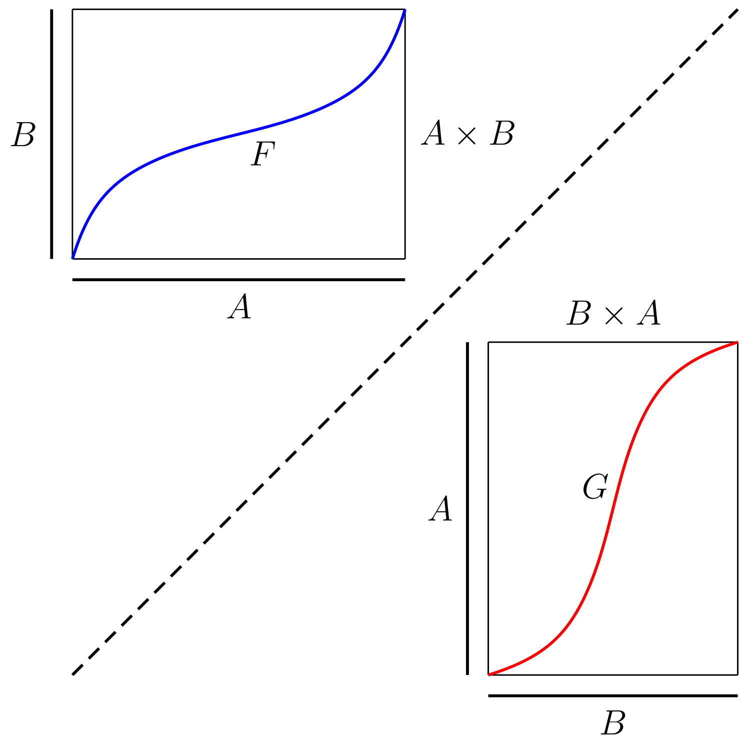
Because of the uniqueness of g as the inverse function of a function f (when such exists), we agree to write g as f^{-1}.

6. **Proof of Theorem (4)**.

Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose f is bijective.

[Ask: How to write down an inverse function of f? What will its graph be?]

Denote by F the graph of f. Define $G = \{(y, x) \mid x \in A \text{ and } y \in B \text{ and } (x, y) \in F\}$.



Define $g = (B, A, G)$.

We have $G \subset B \times A$. So g is a relation.

[We want to verify:

- g is a bijective function.
- g is an inverse function of f.

So recall definitions.]

Here we have a useful observation from definition. The statement $(\sharp)$ holds:

$(\sharp)$: For any $t \in A$, for any $u \in B$, $((t, u) \in F$ iff $(u, t) \in G)$.

Useful Observation. (#): For any $t \in A$, for any $u \in B$, $((t,u) \in F$ iff $(u,t) \in G)$.

Proof of Theorem (4).

... Denote by F the graph of f. Define $G = \{(y, x) \mid x \in A \text{ and } y \in B \text{ and } (x, y) \in F\}$.

Since $f : A \longrightarrow B$ is a bijective function, the following statements hold:

(E): For any $x \in A$, there exists some $y \in B$ such that $(x, y) \in F$.

(U): For any $x \in A$, for any $y, z \in B$, if $(x, y) \in F$ and $(x, z) \in F$ then $y = z$.

(S): For any $y \in B$, there exists some $x \in A$ such that $(x, y) \in F$.

(I): For any $y \in B$, for any $x, w \in A$, if $(x, y) \in F$ and $(w, y) \in F$ then $x = w$.

Consider the relation $g = (B, A, G)$. [We apply (#).]

By (S), (I) respectively, the statements (E'), (U') hold:

(E′): For any $y \in B$, there exists some $x \in A$ such that $(y, x) \in G$.

(U′): For any $y \in B$, for any $x, w \in A$, if $(y, x) \in G$ and $(y, w) \in G$ then $x = w$.

Then g is a function.

By (E), (U) respectively, the statements (S'), (I') hold:

(S′): For any $x \in A$, there exists some $y \in B$ such that $(y, x) \in G$.

(I′): For any $x \in A$, for any $y, z \in B$, if $(y, x) \in G$ and $(z, x) \in G$ then $y = z$

Then g is a bijective function.

[Ask: Is g indeed an inverse function of f?]

Pick any $x \in A$, $y \in B$. Then: $y = f(x)$ iff $(x, y) \in F$ iff $(y, x) \in G$ iff $x = g(y)$.

Therefore g is an inverse function of f. By Theorem (2), it is the unique one. □

7. **Theorem (5)**.

Let A, B, C *be sets and* $f : A \longrightarrow B$, $g : B \longrightarrow C$ *be bijective functions.*

The statements below hold:

(a) $f^{-1} \circ f = \mathsf{id}_A$ *and* $f \circ f^{-1} = \mathsf{id}_B$.

(b) *For any* $x \in A$, *for any* $y \in B$, $y = f(x)$ *iff* $x = f^{-1}(y)$.

(c) f^{-1} *is a bijective function. Moreover,* $(f^{-1})^{-1} = f$.

(d) $g \circ f$ *is a bijective function. Moreover,* $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

Remark. The proof of Theorem (5) is left as an exercise.