MATH1050 'Well-defined-ness' for functions.

0. Refer to the handout *Relations and the formal definition for the notion of functions*.

   Definition. (Relations.)

   Let J, K, L be sets.

   *The ordered triple (J, K, L) is called a **relation from J to K with graph L** if L be a subset of $J \times K$. The sets J, K are respectively called the **set of departure** and the **set of destination** of the relation (J, K, L).*

   Definition. (Functions as relations.)

   Let D, R be sets, and H be a subset of $D \times R$.

   *The relation (D, R, H) is said to be a **function from domain D to range R with graph H** if both of the statements $(E), (U)$ below hold:*

   (E): For any $x \in D$, there exists some $y \in R$ such that $(x, y) \in H$.

   (U): For any $x \in D$, for any $y, z \in R$, if $(x, y) \in H$ and $(x, z) \in H$ then $y = z$.

   Where we refer to (D, R, H) as h, we may write $y = h(x)$ (or $x \xrightarrow{h} y$) exactly when $(x, y) \in H$.

1. **Defining a function by making a 'declaration'.**

   When we define a function, say, f, from A to B, in 'very simple' situations, we often write in this way:

   - *'Define the function $f : A \longrightarrow B$ by $f(x) =$ so-and-so.'*

   What is being done in such a declaration?

   In such a 'declaration', we 'declare' to the reader in the *so-and-so* bit (or what we used to call the 'formula of definition' of the function f in school mathematics) how each element x of A is 'assigned' by the function f to some unique element of B, which we label $f(x)$. In fact, by doing this, we are giving a full description of all the elements of the graph of the function f, albeit not writing the description in set notations.

   This method may lead to pit-falls, if we are not careful enough. The matter concerned is usually referred to as **well-defined-ness of a function**:

   - By saying that $f : A \longrightarrow B$ is **well-defined as a function**, we mean the description of its graph

   $$f(x) = \text{so-and-so} ,$$

   through which we attempt to introduce f to the reader, has been assured to have satisfied Condition (E) and Condition (U).

2. Here we have some 'very simple' situations in which we define various functions by making 'declarations'.

   Examples (A).

   (a) *Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = x^2$ for any $x \in \mathbb{R}$.*

   This tells the reader that each real number is 'assigned' by the function f to its square, which is a uniquely determined real number.

   The graph of the function f is $\{(x, x^2) \mid x \in \mathbb{R}\}$.

   (b) *Define the function $f : \mathbb{R} \longrightarrow \mathbb{R}$ by $f(x) = \begin{cases} -1 & \text{if} \quad x < 0 \\ 0 & \text{if} \quad x = 0 \\ 1 & \text{if} \quad x > 0 \end{cases}$.*

   This tells the reader that each negative real number is 'assigned' by the function f to the real number -1, each positive real number to the real number 1, and the real number 0 to itself.

   The graph of the function f is $\{(x, -1) \mid x < 0\} \cup \{(0, 0)\} \cup \{(y, 1) \mid y > 0\}$.

   (c) *Let $A = \{p, q, r, s, t\}$, $B = \{u, v, w, x, y, z\}$. Define the function $f : A \longrightarrow B$ by $f(p) = f(q) = w$, $f(r) = x$, $f(s) = f(t) = z$.*

   This tells the reader how all the elements of A, namely, p, q, r, s, t, are respectively 'assigned' by the function f to the elements of B. Each of p, q are 'assigned' to w; r to x; each of s, t to z. No element of A is 'left behind'; none is 'assigned' to something 'outside' B.

   The graph of the function f is $\{(p, w), (q, w), (r, x), (s, z), (t, z)\}$.

3. When we define a function by making a 'declaration', we should make sure that Condition (U) is satisfied:

- Each element of the domain of the function to be defined should be assigned to at most one element of the range.

Examples (B).

Which of the below 'declarations' makes sense? Which not? Why?

(a) *Define* $f : \left\{ \dfrac{1}{3}, \dfrac{2}{3}, \dfrac{3}{3} \right\} \longrightarrow \mathbb{R}$ *by* $f(\dfrac{1}{3}) = 1$, $f(\dfrac{2}{3}) = 2$, $f(\dfrac{3}{3}) = 3$.

There is no problem here: f is a function.

(b) *Define* $g : \left\{ \dfrac{p}{q} \ \middle| \ p, q \in [\![1, 3]\!] \right\} \longrightarrow \mathbb{R}$ *by* $g(\dfrac{p}{q}) = p$ *for any* $p, q \in [\![1, 3]\!]$.

It does not make sense to regard g as a function, (although at first sight g looks similar to f):

- Write $A = \left\{ \dfrac{p}{q} \ \middle| \ p, q \in [\![1, 3]\!] \right\}$.

  Note that $A = \left\{ \dfrac{1}{1}, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{2}{1}, \dfrac{2}{2}, \dfrac{2}{3}, \dfrac{3}{1}, \dfrac{3}{2}, \dfrac{3}{3} \right\} = \left\{ 1, \dfrac{1}{2}, \dfrac{1}{3}, 2, \dfrac{2}{3}, 3, \dfrac{3}{2} \right\}$.

  In the above 'declaration', we require g to 'assign' the same element $\dfrac{1}{1}, \dfrac{2}{2}$ of the set A to different elements of $\mathbb{Z}$, namely $1, 2$.

(c) *Define* $h : \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $h(\dfrac{p}{q}) = p$ *whenever* $p \in \mathbb{Z}$ *and* $q \in \mathbb{Z} \backslash \{0\}$.

It does not make sense to regard h as a function:

- In the above 'declaration', we require h to 'assign' at least one element of $\mathbb{Q}$, such as $\dfrac{1}{1}, \dfrac{2}{2}$, to different elements of $\mathbb{Z}$, namely $1, 2$.

4. When we define a function by making a 'declaration', we should make sure that Condition (E) is satisfied:

- Each element of the domain of the function to be defined should be assigned to at least one element of the range.

Examples (C).

Which of the below 'declarations' makes sense? Which not? Why?

(a) *Define* $f : \left\{ \dfrac{1^2}{1^2}, \dfrac{1^2}{2^2}, \dfrac{2^2}{1^2}, \dfrac{2^2}{2^2} \right\} \longrightarrow \mathbb{R}$ *by* $f(\dfrac{1^2}{1^2}) = \dfrac{1}{1}$, $f(\dfrac{1^2}{2^2}) = \dfrac{1}{2}$, $f(\dfrac{2^2}{1^2}) = \dfrac{2}{1}$, $f(\dfrac{2^2}{2^2}) = \dfrac{2}{2}$.

There is no problem here: f is a function.

(b) *Define* $g : \left\{ \dfrac{p}{q} \ \middle| \ p, q \in [\![1, 4]\!] \right\} \longrightarrow \mathbb{R}$ *by* $g(\dfrac{s^2}{t^2}) = \dfrac{s}{t}$, *whenever* $s^2, t^2 \in [\![1, 4]\!]$ *and* $s, t \in \mathbb{N}$.

It does not make sense to regard g as a function, (although at first sight g looks similar to f):

- Write $A = \left\{ \dfrac{p}{q} \ \middle| \ p, q \in [\![1, 4]\!] \right\}$. Note that $A = \left\{ 1, \dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, 2, \dfrac{2}{3}, 3, \dfrac{3}{2}, \dfrac{3}{4}, 4, \dfrac{4}{3} \right\}$.

  In the above 'declaration', we fail to require g to 'assign' some element of A, such as $\dfrac{2}{3}$, to some element of $\mathbb{Z}$.

(c) *Define* $h : \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $h(r^2) = r$ *whenever* $r \in \mathbb{Q}$.

It does not make sense to regard h as a function:

- In the above 'declaration', we fail to require h to 'assign' at least one element of $\mathbb{Q}$, such as $\dfrac{2}{3}$, to some element of $\mathbb{R}$.

5. **Examples (D).**

Which of the below 'declarations' makes sense as a 'formula of definition for a function'? Which not? Why?

(a) *Define* $f : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $f(z) = |z|$ *for any* $z \in \mathbb{C}$.

There is no problem here: f is a function.

(b) *Define* $g : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $g(z) = i|z|$ *for any* $z \in \mathbb{C}$.

It does not make sense to regard g as a function, (although at first sight g looks similar to f):

- The range of the function we intend to define is $\mathbb{R}$. However, in the above 'declaration', we require g to 'assign' some element of $\mathbb{R}$, such as 1, to the number i, which is outside $\mathbb{R}$.

(c) Define $h : \mathbb{C} \longrightarrow \mathbb{R}$ by $h(z) = \dfrac{z^4 \bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1}$ for any $z \in \mathbb{C}$.

h is indeed **well-defined as a function** from $\mathbb{C}$ to $\mathbb{R}$. But it is not apparent until we have verified that for each $z \in \mathbb{C}$, $\dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1}$ is well-defined as a real number.

Appropriate work should precede or follow the above 'declaration' to assure the reader that it makes sense to regard h as a function. Below is an illustration on how this is done:

> We claim that for any $z \in \mathbb{C}$, $2|z|^2 + z^2 + (\bar{z})^2 + 1 \neq 0$ and $\dfrac{z^4\bar{z}+iz^3(\bar{z})^2-iz^2(\bar{z})^3+z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1} \in \mathbb{R}$.
>
> Justification: Let $z \in \mathbb{C}$. $2|z|^2 + z^2 + (\bar{z})^2 + 1 = ... \neq 0$. Et cetera.
>
> Define the function $h : \mathbb{C} \longrightarrow \mathbb{R}$ by $h(z) = \dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1}$ for any $z \in \mathbb{C}$.

So, by the time we write down what h 'does', we have already made sure that no element of $\mathbb{C}$ is 'assigned' by h to something outside the intended range of h.

6. It may happen that when we define a function by making a 'declaration', it is neither immediately clear that Condition (E) is satisfied nor immediately clear that Condition (U) is satisfied.

In this situation, rather than 'declaring' through a 'formula of definition' what the function to be defined 'does', it may be better to first express explicitly in set notations the graph of the function to be defined and then verify that Condition (E) and Condition (U) are satisfied.

Examples (E).

Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Here we wonder whether the below 'declaration' makes sense:

- Define $g : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ by $g(z) = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta))$ whenever $z \in \mathbb{C}^*$ and θ is an argument of z.

Note that if g is a function then its graph is given by the set

$$\left\{(z,\zeta) \;\middle|\; \begin{array}{l} z, \zeta \in \mathbb{C}^* \text{ and there exists some } \theta \in \mathbb{R} \text{ such that} \\ z = |z|(\cos(\theta) + i\sin(\theta)) \text{ and } \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) \end{array}\right\}.$$

We proceed to check that g is well-defined as a function below:

> Define the subset G of $\mathbb{C}^* \times \mathbb{C}^*$ by
>
> $$G = \left\{(z,\zeta) \;\middle|\; \begin{array}{l} z, \zeta \in \mathbb{C}^* \text{ and there exists some } \theta \in \mathbb{R} \text{ such that} \\ z = |z|(\cos(\theta) + i\sin(\theta)) \text{ and } \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) \end{array}\right\}.$$
>
> Define $g = (\mathbb{C}^*, \mathbb{C}^*, G)$.
>
> * [Does g satisfy Condition (E)?]
>   Pick any $z \in \mathbb{C}^*$. There exists some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta) + i\sin(\theta))$.
>   Define $\zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta))$.
>   Since $z \in \mathbb{C}^*$, $\sqrt{|z|} \neq 0$. Also note that $|\cos(2\theta) + i\sin(2\theta)| = 1$. Then $\zeta \in \mathbb{C}^*$.
>   By definition, we have $(z,\zeta) \in G$.
> * [Does g satisfy Condition (U)?]
>   Pick any $z \in \mathbb{C}^*$. Pick any $\zeta, \omega \in \mathbb{C}^*$. Suppose $(z,\zeta) \in G$ and $(z,\omega) \in G$.
>   Since $(z,\zeta) \in G$, there exist some $\theta \in \mathbb{R}$ such that $z = |z|(\cos(\theta)+i\sin(\theta))$ and $\zeta = \sqrt{|z|}(\cos(2\theta)+i\sin(2\theta))$.
>   Since $(z,\omega) \in G$, there exist some $\varphi \in \mathbb{R}$ such that $z = |z|(\cos(\varphi)+i\sin(\varphi))$ and $\omega = \sqrt{|z|}(\cos(2\varphi)+i\sin(2\varphi))$.
>   Note that $|z|(\cos(\theta) + i\sin(\theta)) = z = |z|(\cos(\varphi) + i\sin(\varphi))$. Since $|z| \neq 0$, we have $\cos(\theta) = \cos(\varphi)$ and $\sin(\theta) = \sin(\varphi)$.
>   Then there exists some $k \in \mathbb{Z}$ such that $\theta = \varphi + 2k\pi$. Therefore
>
> $$\begin{aligned} \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) &= \sqrt{|z|}(\cos(2(\theta + 2k\pi)) + i\sin(2(\theta + 2k\pi))) \\ &= \sqrt{|z|}(\cos(2\varphi) + i\sin(2\varphi)) = \omega. \end{aligned}$$

> It follows that $g : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ is a function, whose 'formula of definition' is given by $g(z) = \sqrt{|z|}(\cos(2\theta)+i\sin(2\theta))$ whenever $z \in \mathbb{C}^*$ and θ is an argument of z.