0. Refer to the handout *Relations and the formal definition for the notion of functions.*

Definition. (Relations.)

Let J, K, L be sets.

The ordered triple (J, K, L) is called a **relation from J to K with graph** *L if L be a subset of $J \times K$.*

The sets J, K are respectively called the **set of departure** *and the* **set of destination** *of the relation (J, K, L).*

Definition. (Functions as relations.)

Let D, R be sets, and H be a subset of $D \times R$.

The relation (D, R, H) is said to be a **function from domain D to range R with graph** *H if both of the statements $(E), (U)$ below hold:*

(E): *For any $x \in D$, there exists some $y \in R$ such that $(x, y) \in H$.*

(U): *For any $x \in D$, for any $y, z \in R$, if $(x, y) \in H$ and $(x, z) \in H$ then $y = z$.*

Where we refer to (D, R, H) as h, we may write $y = h(x)$ (or $x \overset{}{\underset{h}{\longmapsto}} y$) exactly when $(x, y) \in H$.

1. **Defining a function by making a 'declaration'.**

When we define a function, say, f, from A to B, in 'very simple' situations, we often write in this way:

- *'Define the function* $f : A \longrightarrow B$ *by* $f(x) =$ *so-and-so.'*

What is being done in such a declaration?

In such a 'declaration', we 'declare' to the reader in the ***so-and-so*** bit (or what we used to call the 'formula of definition' of the function f in school mathematics) how each element x of A is 'assigned' by the function f to some unique element of B, which we label $f(x)$.

We are giving a full description of all the elements of the graph of the function f, albeit not writing the description in set notations.

This method may lead to pit-falls, if we are not careful enough. The matter concerned is usually referred to as **well-defined-ness of a function**:

- By saying that $f : A \longrightarrow B$ is **well-defined as a function**, we mean the description of its graph

$$f(x) = \textit{so-and-so} \; ,$$

through which we attempt to introduce f to the reader, has been assured to have satisfied Condition (E) and Condition (U).

2. **Examples (A)**.

(a) *Define the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(x) = x^2$ *for any* $x \in \mathbb{R}$.

Graph of the function f?
$\{(x, x^2) \mid x \in \mathbb{R}\}$

(b) *Define the function* $f : \mathbb{R} \longrightarrow \mathbb{R}$ *by* $f(x) = \begin{cases} -1 & \text{if } x < 0 \\ 0 & \text{if } x = 0 \\ 1 & \text{if } x > 0 \end{cases}$.

Graph of the function f?
$\{(s,-1) \mid s<0\} \cup \{(0,0)\} \cup \{(t,1) \mid t>0\}$

(c) *Let* $A = \{p, q, r, s, t\}$, $B = \{u, v, w, x, y, z\}$.
Define the function $f : A \longrightarrow B$ *by* $f(p) = f(q) = w$, $f(r) = x$, $f(s) = f(t) = z$.

3. **Examples (B)**.

Which of the below 'declarations' makes sense? Which not? Why?

(a) *Define* $f : \left\{\frac{1}{3}, \frac{2}{3}, \frac{3}{3}\right\} \longrightarrow \mathbb{R}$ *by* $f(\frac{1}{3}) = 1$, $f(\frac{2}{3}) = 2$, $f(\frac{3}{3}) = 3$.

No problem. f is a function.

(b) *Define* $g : \left\{ \frac{p}{q} \;\middle|\; p, q \in [\![1, 3]\!] \right\} \longrightarrow \mathbb{R}$ *by* $g(\frac{p}{q}) = p$ *for any* $p, q \in [\![1, 3]\!]$.

It does not make sense to regard g as a function.

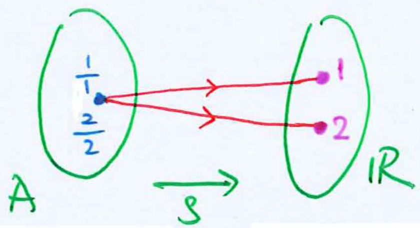
- Write $A = \left\{ \frac{p}{q} \;\middle|\; p, q \in [\![1, 3]\!] \right\}$.

$A = \left\{ \frac{1}{1}, \frac{1}{2}, \frac{1}{3}, \frac{2}{1}, \frac{2}{2}, \frac{2}{3}, \frac{3}{1}, \frac{3}{2}, \frac{3}{3} \right\} = \left\{ 1, \frac{1}{2}, \frac{1}{3}, 2, \frac{2}{3}, 3, \frac{3}{2} \right\}$.

g assigns $\frac{1}{1}$ to 1.

g assigns $\frac{2}{2}$ to 2.

But $\frac{1}{1} = \frac{2}{2} = 1$.



(c) *Define* $h : \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $h(\frac{p}{q}) = p$ *whenever* $p \in \mathbb{Z}$ *and* $q \in \mathbb{Z}\backslash\{0\}$.

It does not make sense to regard h as a function.

h assigns $\frac{1}{1}$ to 1. h assigns $\frac{2}{2}$ to 2. But $\frac{1}{1} = \frac{2}{2} = 1$.

4. **Examples (C).**

Which of the below 'declarations' makes sense? Which not? Why?

(a) *Define* $f : \left\{ \frac{1^2}{1^2}, \frac{1^2}{2^2}, \frac{2^2}{1^2}, \frac{2^2}{2^2} \right\} \longrightarrow \mathbb{R}$ *by* $f(\frac{1^2}{1^2}) = \frac{1}{1}$, $f(\frac{1^2}{2^2}) = \frac{1}{2}$, $f(\frac{2^2}{1^2}) = \frac{2}{1}$, $f(\frac{2^2}{2^2}) = \frac{2}{2}$.

No problem. f is a function.

(b) *Define* $g : \left\{ \frac{p}{q} \;\middle|\; p, q \in [\![1, 4]\!] \right\} \longrightarrow \mathbb{R}$ *by* $g(\frac{s^2}{t^2}) = \frac{s}{t}$, *whenever* $s^2, t^2 \in [\![1, 4]\!]$ *and* $s, t \in \mathbb{N}$.

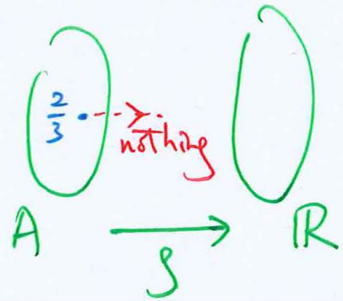
It does not make sense to regard g as a function.

- Write $A = \left\{ \frac{p}{q} \;\middle|\; p, q \in [\![1, 4]\!] \right\}$.

  $A = \left\{ 1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, 2, \frac{2}{3}, 3, \frac{3}{2}, \frac{3}{4}, 4, \frac{4}{3} \right\}$.

  For any $s, t \in \mathbb{N}$, if $s^2, t^2 \in [\![1, 4]\!]$ then $\frac{s^2}{t^2} \neq \frac{2}{3}$.

  g does not assign $\frac{2}{3}$ to any element of $\mathbb{R}$.



(c) *Define* $h : \mathbb{Q} \longrightarrow \mathbb{R}$ *by* $h(r^2) = r$ *whenever* $r \in \mathbb{Q}$.

It does not make sense to regard h as a function.

$\frac{2}{3} \in \mathbb{Q}$. But, for any $r \in \mathbb{Q}$, $r^2 \neq \frac{2}{3}$.

h does not assign $\frac{2}{3}$ to any element of $\mathbb{R}$.

5. **Examples (D).**

Which of the below 'declarations' makes sense as a 'formula of definition for a function'? Which not? Why?

(a) *Define* $f : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $f(z) = |z|$ *for any* $z \in \mathbb{C}$.

No problem. f is a function.

(b) *Define* $g : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $g(z) = i|z|$ *for any* $z \in \mathbb{C}$.

It does not make sense to regard g as a function.

$1 \in \mathbb{C}$. $i \cdot |1| = i \notin \mathbb{R}$.

g assigns $1 \in \mathbb{C}$ to something outside $\mathbb{R}$, namely i.

(c) *Define* $h : \mathbb{C} \longrightarrow \mathbb{R}$ *by* $h(z) = \dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1}$ *for any* $z \in \mathbb{C}$.

h is well-defined as a function, but it has to be justified:

- Claim: For any $z \in \mathbb{C}$, $2|z|^2 + z^2 + (\bar{z})^2 + 1 \neq 0$.
- Further claim: For any $z \in \mathbb{C}$, $\dfrac{z^4\bar{z} + iz^3(\bar{z})^2 - iz^2(\bar{z})^3 + z(\bar{z})^4}{2|z|^2 + z^2 + (\bar{z})^2 + 1} \in \mathbb{R}$.

6. **Examples (E).**

Write $\mathbb{C}^* = \mathbb{C}\backslash\{0\}$. Here we wonder whether the below 'declaration' makes sense:

- *Define* $g : \mathbb{C}^* \longrightarrow \mathbb{C}^*$ *by* $g(z) = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta))$ *whenever* $z \in \mathbb{C}^*$ *and* θ *is an argument of* z.

Note that if g is a function then its graph is given by the set

$$\left\{ (z,\zeta) \;\middle|\; \begin{array}{l} z,\zeta \in \mathbb{C}^* \text{ and there exists some } \theta \in \mathbb{R} \text{ such that} \\ z = |z|(\cos(\theta) + i\sin(\theta)) \text{ and } \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) \end{array} \right\}.$$

We proceed to check that g is well-defined as a function below:

Define the subset G *of* $\mathbb{C}^* \times \mathbb{C}^*$ *by*

$$G = \left\{ (z,\zeta) \;\middle|\; \begin{array}{l} z,\zeta \in \mathbb{C}^* \textit{ and there exists some } \theta \in \mathbb{R} \textit{ such that} \\ z = |z|(\cos(\theta) + i\sin(\theta)) \textit{ and } \zeta = \sqrt{|z|}(\cos(2\theta) + i\sin(2\theta)) \end{array} \right\}.$$

Define $g = (\mathbb{C}^*, \mathbb{C}^*, G)$.

$*$ *[Does* g *satisfy Condition (E)?]*

$*$ *[Does* g *satisfy Condition (U)?]*