MATH1050 Relations and the formal definition for the notion of functions.

1. **'In-formal' definition for the notion of function**.

   Recall the in-formal definition for the notion of function':

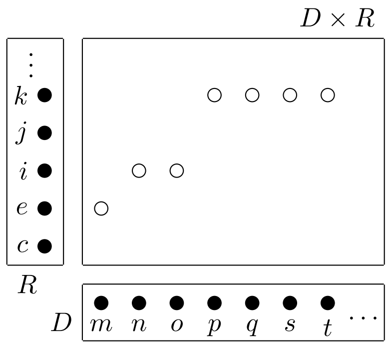
   Let D, R be sets. h is a function from D to R exactly when h is a 'rule of assignment' from D to R, so that

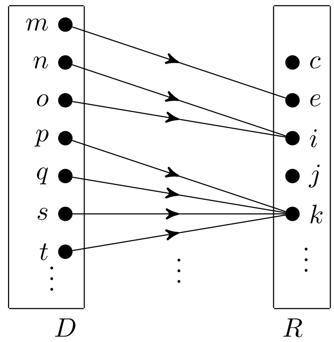
   > *each element x of D is being assigned to exactly one element, namely $h(x)$, of R.*

   D is called the domain of h. R is called the range of h.

   Below are the 'coordinate plane diagram' and the 'blobs-and-arrow diagram' for such a mathematical object, say, the function $h : D \longrightarrow R$, where:

   - $D = \{m, n, o, p, q, s, t, ...\}$, $R = \{c, e, i, j, k, ...\}$,
   - $h(m) = e, h(n) = i, h(o) = i, h(p) = k, h(q) = k, h(s) = k, h(t) = k, \cdots$, and
   - The graph of the function h is the set $H = \{(m, e), (n, i), (o, i), (p, k), (q, k), (s, k), (t, k), \cdots\}$.





2. **Problem in the 'in-formal' definition for the notion of function, and the solution**.

   The problem with the above 'in-formal definition' is that it is not clear what we mean by the phrase '*rule of assignment*', which is crucial in explaining the mathematical meaning of the word 'function'.

   To solve this problem, we try to formulate an appropriate definition for the notion of function in terms of something that we understand better mathematically: we shall write in terms of the language of sets.

   But what kind of sets shall we be looking at?

   In order to understand what any particular function does as a 'rule of assignment' from its domain and its range, we study its graph, which is a subset of the cartesian product of the domain and the range. In fact the graph of the function contains exactly all the information about that function: we can recover from its graph what the function does as a 'rule of assignment' from its domain to its range.

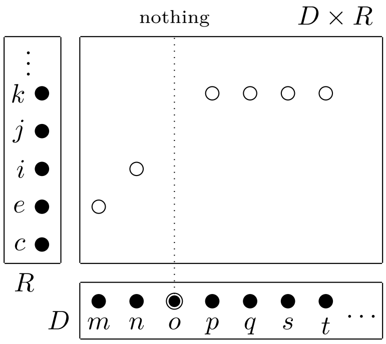
   Hence we focus on how to make sense of '*graph of a function*' in terms of the language of sets, instead of making sense of '*rule of assignment*'.

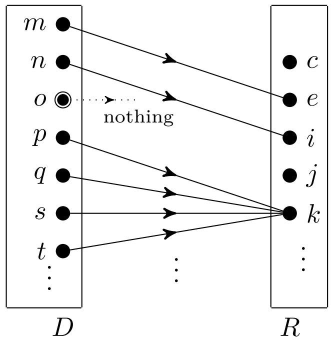
3. **Towards the formal definition for the notion of function**.

   We expect the graph of a function to be necessarily a subset of the cartesian product of the domain and the range. However, it cannot be just any subset: it has to satisfy appropriate conditions so as to make sense of '*each element of the domain of the function is being assgined to exactly one element of the range of the function*', which amounts to the upholding of two kinds of forbiddance:

   (a) **'First forbiddance'**:

   We forbid a mathematical object from being called a function from D to R if it happens that, were it a function, its 'coordinate plane diagram' and 'blobs-and-arrows diagram' look like something as below.

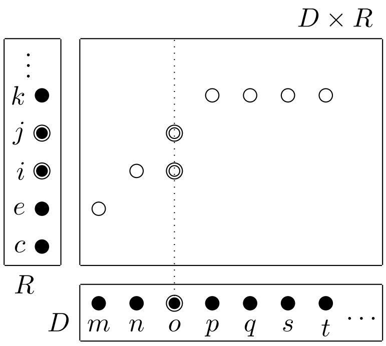


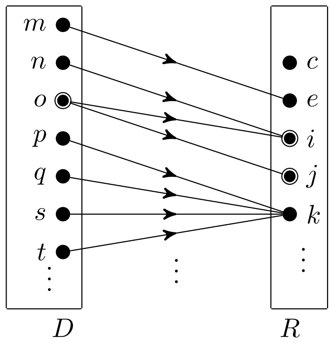


The reason for this forbiddance is that some element of D, namely o, is being assigned to no element of R.

(b) **'Second forbiddance'**:

We forbid a mathematical object from being called a function from D to R if it happens that, were it a function, its 'coordinate plane diagram' and 'blobs-and-arrows diagram' look like something as below.





The reason for this forbiddance is that some element of D, namely o, is being assigned to distinct elements of R, namely i, j.

With these two kinds of forbiddance in mind, we formulate the formal definition for the notion of functions, in terms of that for relations.

4. **Definition. (Relations.)**

Let J, K, L be sets.

The ordered triple (J, K, L) is called a **relation from J to K with graph** L *if L be a subset of $J \times K$. The sets J, K are respectively called the* **set of departure** *and the* **set of destination** *of the relation (J, K, L).*

Definition. (Functions as relations.)

Let D, R be sets, and H be a subset of $D \times R$.

The relation (D, R, H) is said to be a **function from domain D to range R with graph** H *if both of the statements $(E), (U)$ below hold:*

(E): *For any $x \in D$, there exists some $y \in R$ such that $(x, y) \in H$.*

(U): *For any $x \in D$, for any $y, z \in R$, if $(x, y) \in H$ and $(x, z) \in H$ then $y = z$.*

Where we refer to (D, R, H) as h, we may write $y = h(x)$ (or $x \underset{h}{\longmapsto} y$) exactly when $(x, y) \in H$.

Remarks.

(a) It is through the graph H of the function h that we understand how h assigns the elements of its domain D to its range R. Condition (E) and Condition (U) are formulated to describe what we want H to satisfy as a subset of $D \times R$.

(b) In plain words, When Conditions (E), (U) read:

(E): *Each element of D is assigned by h to at least one element of R.*

(U): *Each element of D is assigned by h to at most one element of R.*

So Condition (E), (U) respectively guarantee that the 'first forbiddance' and the 'second forbiddance' are upheld.

(c) The conjunction '(E) and (U)' reads:

(EU): *Each element of D is assigned by h to exactly one element of R.*

Thus we have 'recovered' the 'in-formal definition for the notion of function'.