1. **‘In-formal’ definition for the notion of function**.

Recall the in-formal definition for the notion of function’:

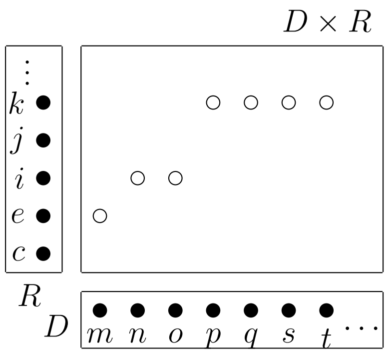
Let D, R be sets.

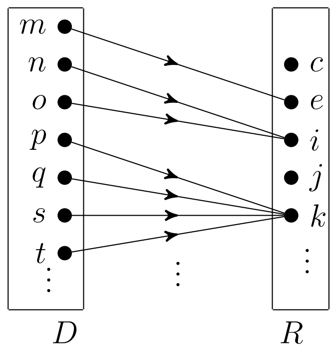
h is a function from D to R exactly when h is a ‘rule of assignment’ from D to R, so that

each element x of D is being assigned to exactly one element, namely $h(x)$, of R.

D is called the domain of h. R is called the range of h.

Below are the ‘coordinate plane diagram’ and the ‘blobs-and-arrow diagram’ for such a mathematical object, say, the function $h : D \longrightarrow R$.





Here $D = \{m, n, o, p, q, s, t, ...\}$, $R = \{c, e, i, j, k, ...\}$,
$h(m) = e, h(n) = i, h(o) = i, h(p) = k, h(q) = k, h(s) = k, h(t) = k, \cdots$, and
the graph of h is the set $H = \{(m, e), (n, i), (o, i), (p, k), (q, k), (s, k), (t, k), \cdots\}$.

2. **Problem in the 'in-formal' definition for the notion of function, and the solution.**

- What do we mean by the phrase '*rule of assignment*'?

The meaning of this phrase is un-clear.
But this phrase is crucial in explaining the mathematical meaning of the word 'function'.

- How to solve this problem?

Formulate an appropriate definition for the notion of function in terms of something that we understand better mathematically: language of sets.

- But what kind of sets shall we be looking at? Why?

The graph of a function is a subset of the cartesian product of the domain and the range of the function.
This carries all the information of the function as a 'rule of assignment' from its domain to its range.
So we focus on making sense of the phrase 'graph of a function'.

3. **Towards the formal definition for the notion of function**.

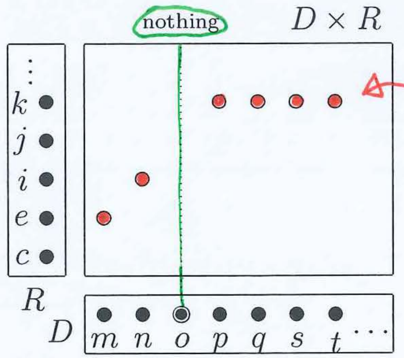
We expect the graph of a function to be necessarily a subset of the cartesian product of the domain and the range.

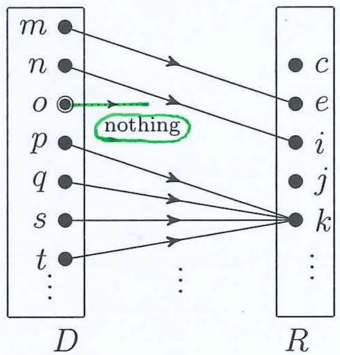
However, it cannot be just any subset:

As a 'rule of assignment', the function has to satisfy:
each element of its domain is being assigned to exactly one element of its range.

(a) **'First forbiddance'**:

Subsets of $D \times R$ like the one below will not be allowed to be the graph of any function:





The reason for this forbiddance is that some element of D, namely o, is being assigned to no element of R.

To be more precise?

[Write $G = \{(m,e), (n,i), (p,k), (q,k), (s,k), (t,k), \ldots\}$.
For this G, there exists some $x \in D$, namely $x = o$, such that for any $y \in R$, $(x,y) \notin G$.]

Towards the formal definition for the notion of function.

We expect the graph of a function to be necessarily a subset of the cartesian product of the domain and the range.

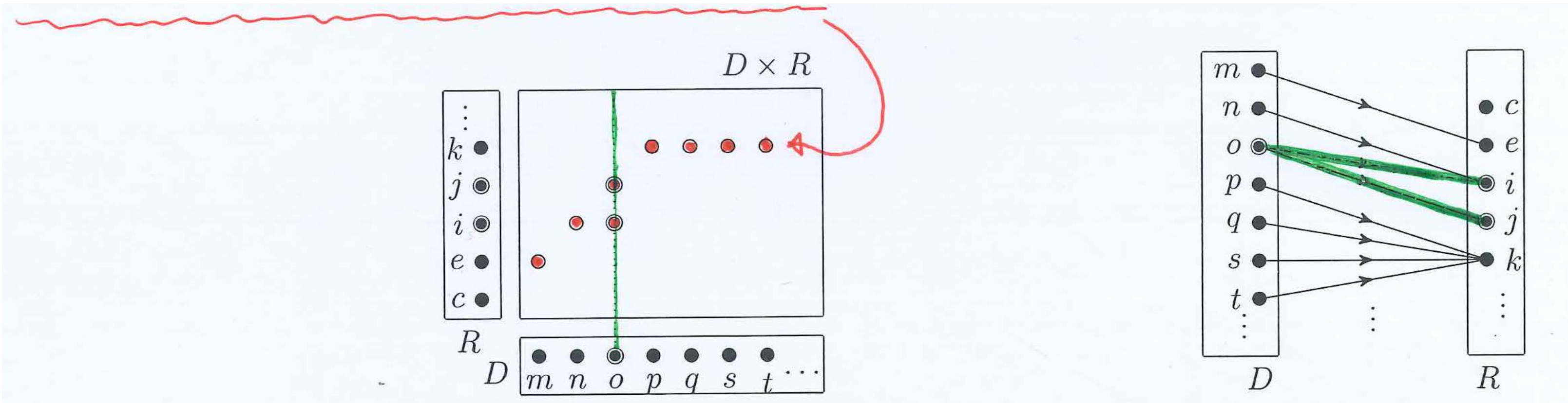
However, it cannot be just any subset:

As a 'rule of assignment', the function has to satisfy:
each element of its domain is being assigned to exactly one element of its range.

(a) 'First forbiddance': ...

(b) **'Second forbiddance'**:

Subsets of $D \times R$ like the one below will not be allowed to be the graph of any function:



The reason for this forbiddance is that some element of D, namely o, is being assigned to distinct elements of R, namely i, j.

To be more precise?

Write $G = \{(m,e), (n,i), (o,i), (o,j), (p,k), (q,k), (s,k), (t,k), \ldots\}$

For this G, there exits some $x \in D$, $y, z \in R$, namely $x = o$, $y = i$, $z = j$, such that $(x,y) \in G$ and $(x,z) \in G$ and $y \neq z$.

4. **Definition. (Relations.)**

Let J, K, L be sets.

The ordered triple (J, K, L) is called a **relation from J to K with graph** *L if L be a subset of $J \times K$.*

The sets J, K are respectively called the **set of departure** *and the* **set of destination** *of the relation (J, K, L).*

Definition. (Functions as relations.)

Let D, R be sets, and H be a subset of $D \times R$.

The relation (D, R, H) is said to be a **function from domain D to range R with graph** *H if both of the statements $(E), (U)$ below hold:*

(E): *For any $x \in D$, there exists some $y \in R$ such that $(x, y) \in H$.*

(U): *For any $x \in D$, for any $y, z \in R$, if $(x, y) \in H$ and $(x, z) \in H$ then $y = z$.*

Where we refer to (D, R, H) as h, we may write $y = h(x)$ (or $x \underset{h}{\longmapsto} y$) exactly when $(x, y) \in H$.

Definition. (Functions as relations.)

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The relation (D, R, H) is said to be a **function from domain D to range R with graph H** *if both of the statements $(E), (U)$ below hold:*

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(U): *For any $x \in D$, for any $y, z \in R$, if $(x, y) \in H$ and $(x, z) \in H$ then $y = z$.*

Remarks.

(a) It is through the graph H of the function h that we understand how h assigns the elements of its domain D to its range R.

Condition (E) and Condition (U) are formulated to describe what we want H to satisfy as a subset of $D \times R$.

(b) In plain words, When Conditions (E), (U) read:

(E): *Each element of D is assigned by h to at least one element of R.*

(U): *Each element of D is assigned by h to at most one element of R.*

So Condition (E), (U) respectively guarantee that the 'first forbiddance' and the 'second forbiddance' are upheld.

(c) The conjunction '(E) and (U)' reads:

(EU): *Each element of D is assigned by h to exactly one element of R.*

Thus we have 'recovered' the 'in-formal definition for the notion of function'.