1. **Definition.**
Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions. g is said to be an **inverse function** *of f if both of the following statements hold:*

   (a) *For any $x \in A$, $(g \circ f)(x) = x$.*

   (b) *For any $y \in B$, $(f \circ g)(y) = y$.*

   Definition.
Let C be a set. Define the function $\mathsf{id}_C : C \longrightarrow C$ by $\mathsf{id}_C(z) = z$ for any $z \in C$. id_C is called the **identity function on the set** *C.*

   Remark 1 on the definition for the notion of inverse function.
By the respective definitions for the notions of inverse function, composition, and identity function:

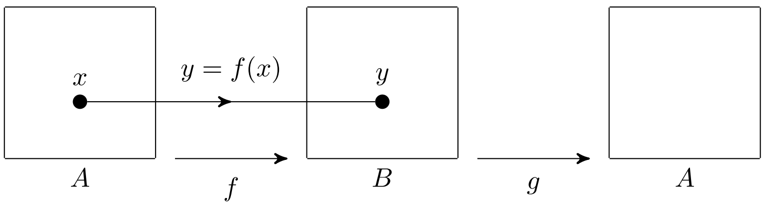
   $g : B \longrightarrow A$ is an inverse function of $f : A \longrightarrow B$ iff ($g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$ as functions).

   Remark 2 on the definition for the notion of inverse function.
Note the 'symmetry' in the definition for the notion of inverse function.
A consequence of this 'symmetry' is:

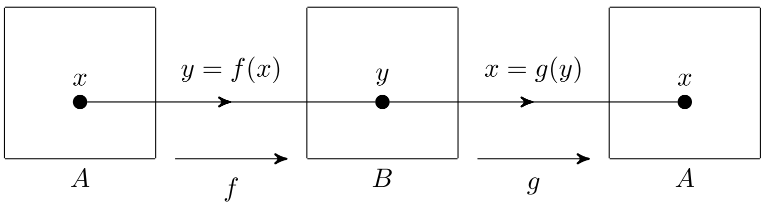
   $g : B \longrightarrow A$ is an inverse function of $f : A \longrightarrow B$ iff $f : A \longrightarrow B$ is an inverse function of $g : B \longrightarrow A$.

   Remark 3 on the definition for the notion of inverse function.
How does such a function g 'interact' with f? (First recall the notion of composition of functions.)

   (a) Pick any $x \in A$. x is 'assigned' by f to $f(x)$.

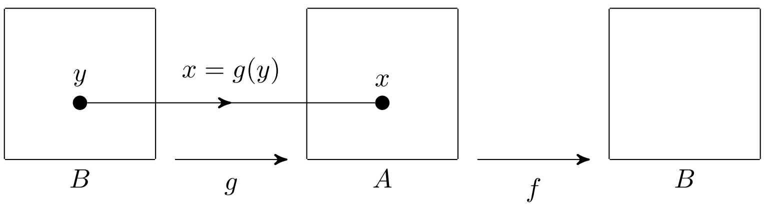


   But then $f(x)$ is 'assigned' by g to x.

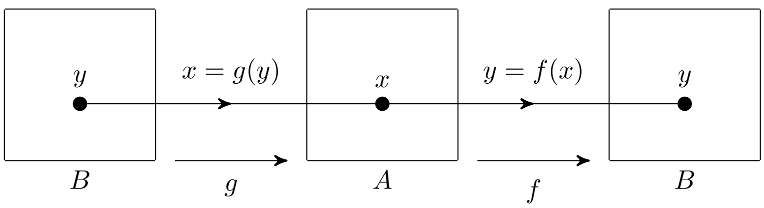


   So g 'cancels' what f does to x.
This is a formal way to tell the above story: *for any $x \in A$, for any $y \in B$, if $y = f(x)$ then $x = g(y)$.*

   (b) Pick any $y \in B$. y is 'assigned' by g to $g(y)$.



   But then $g(y)$ is 'assigned' by f to y.



   So f 'cancels' what g does to y.
This is a formal way to tell the above story: *for any $y \in B$, for any $x \in A$, if $x = g(y)$ then $y = f(x)$.*

   We may combine the above as: *for any $x \in A$, for any $y \in B$, ($y = f(x)$ iff $x = g(y)$).*

2. **Theorem (1). (Re-formulation of the definition of inverse function.)**

Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions. The statements below are logically equivalent:

($\star_0$) *g is an inverse function of f.*

($\star_1$) *$g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$ as functions.*

($\star_2$) *f is an inverse function of g.*

($\star_3$) *For any $x \in A$, for any $y \in B$, ($y = f(x)$ iff $x = g(y)$).*

Proof of Theorem (1).

Let A, B be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow A$ be functions.

By definition, the statements $(\star_0), (\star_1), (\star_2)$ are logically equivalent:

($\star_0$) g is an inverse function of f. ($\star_1$) $g \circ f = \mathsf{id}_A$ and $f \circ g = \mathsf{id}_B$. ($\star_2$) f is an inverse function of g.

We are going to verify that the statements $(\star_0), (\star_3)$ are logically equivalent:

($\star_0$) g is an inverse function of f. ($\star_3$) For any $x \in A$, for any $y \in B$, ($y = f(x)$ iff $x = g(y)$).

- [$(\star_0) \Longrightarrow (\star_3)$?]
  Suppose g is an inverse function of f. Pick any $x \in A$, $y \in B$.
  - Suppose $y = f(x)$. Then $g(y) = g(f(x)) = (g \circ f)(x) = x$ by definition of inverse function.
  - Suppose $x = g(y)$. Then $f(x) = f(g(y)) = (f \circ g)(y) = y$ by definition of inverse function.

  It follows that $y = f(x)$ iff $x = g(y)$.
- [$(\star_3) \Longrightarrow (\star_0)$?]
  Suppose that for any $x \in A$, $y \in B$, ($y = f(x)$ iff $x = g(y)$).
  - Pick any $s \in A$. Define $u = f(s)$. We have $u \in B$. By assumption $s = g(u)$. Then $(g \circ f)(s) = g(f(s)) = g(u) = s$.
  - Pick any $v \in B$. Define $t = g(v)$. We have $t \in A$. By assumption $v = f(t)$. Then $(f \circ g)(v) = f(g(v)) = f(t) = v$.

  It follows that g is an inverse function of f.

3. **Theorem (2). (Uniqueness of inverse function.)**

Let A, B be sets, and $f : A \longrightarrow B$ be a function. f has at most one inverse function.

Proof of Theorem (2).

Let A, B be sets, and $f : A \longrightarrow B$ be a function. Suppose $g, h : B \longrightarrow A$ are inverse functions of f.

[We want to deduce that $g(y) = h(y)$ for any $y \in B$.]

Pick any $y \in B$. Define $x = g(y)$. We have $x \in A$. Then $y = f(g(y)) = f(x)$. Therefore $h(y) = h(f(x)) = x = g(y)$.

It follows that g, h are the same function.

4. **Definition.**

Let D, R be sets and $h : D \longrightarrow R$ be a function. h is said to be **bijective** *if h is both surjective and injective.*

Remark. Hence h is bijective iff both of the statements $(S), (I)$ below hold:

(S): *For any $v \in R$, there exists some $u \in D$ such that $v = h(u)$.*

(I): *For any $u, t \in D$, if $h(u) = h(t)$ then $u = t$.*

5. **Theorem (3). (Necessary condition for existence of inverse function.)**

Let A, B be sets, $f : A \longrightarrow B$ be a function. Suppose f has an inverse function, say, $g : B \longrightarrow A$. Then each of f, g is bijective.

Proof of Theorem (3).

Let A, B be sets, $f : A \longrightarrow B$ be a function. Suppose f has an inverse function, say, $g : B \longrightarrow A$.

- [Ask: 'Is f surjective?']
  Pick any $y \in B$. Define $x = g(y)$. We have $x \in A$. For the same x, y, we have $f(x) = f(g(y)) = y$. Therefore f is surjective.
- [Ask: 'Is f injective?']
  Pick any $x, w \in A$. Suppose $f(x) = f(w)$. Then $x = g(f(x)) = g(f(w)) = w$. Therefore f is injective.

By definition, g is an inverse function of f. Then by Theorem (1), g has an inverse function, namely, f. It follows from the argument above that g is both surjective and injective.

Remark. The natural question to ask is: *Is the necessary condition sufficient?*