1. **Definitions**.

(a) *Let* A, B, C *be sets, and* $f : A \longrightarrow B$, $g : B \longrightarrow C$ *be functions.*
Define the function $g \circ f : A \longrightarrow C$ *by* $(g \circ f)(x) = g(f(x))$ *for any* $x \in A$.
$g \circ f$ *is called the* **composition** *of the functions* f, g.

(b) *Let* D, R *be sets, and* $h : D \longrightarrow R$ *be a function.*

i. h *is said to be* **surjective** *if*

$$(\text{for any } v \in R \text{ there exists some } u \in D \text{ such that } v = h(u)).$$

ii. h *is said to be* **injective** *if*

$$(\text{for any } t, u \in D, \text{ if } h(t) = h(u) \text{ then } t = u).$$

2. **Theorem** $(\natural_1)$.

Let A, B, C *be sets, and* $f : A \longrightarrow B$, $g : B \longrightarrow C$ *be functions.*
The following statements hold:

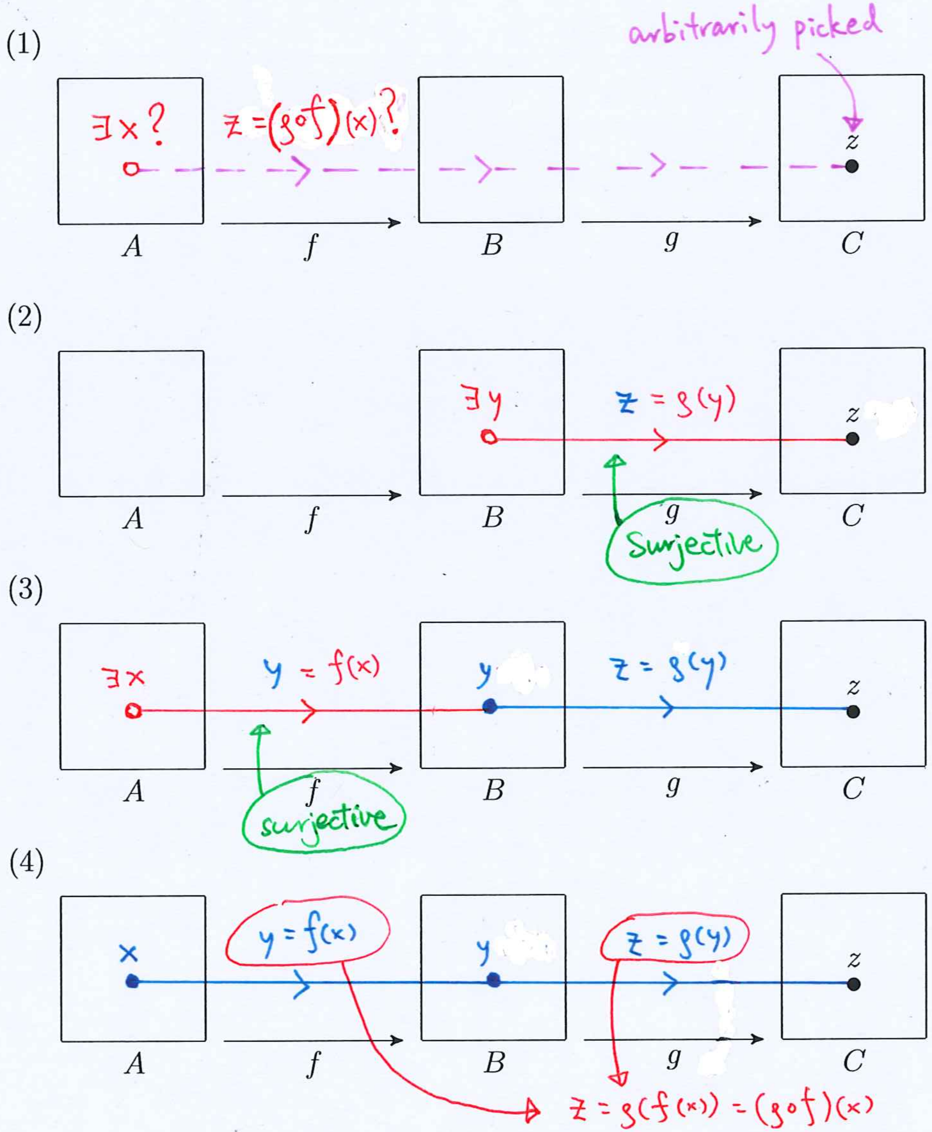
(1) *Suppose* f, g *are surjective. Then* $g \circ f$ *is surjective.*

(2) *Suppose* f, g *are injective. Then* $g \circ f$ *is injective.*

3. **Proof of Statement (1) of Theorem ($\sharp_1$)** .

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

Suppose f, g are surjective. [We want to verify that $g \circ f$ is surjective under this assumption.]



For any $z \in C$,
there exists some $x \in A$ such that $z = (g \circ f)(x)$.

Appropriate argument should read as:

Pick any $z \in C$.

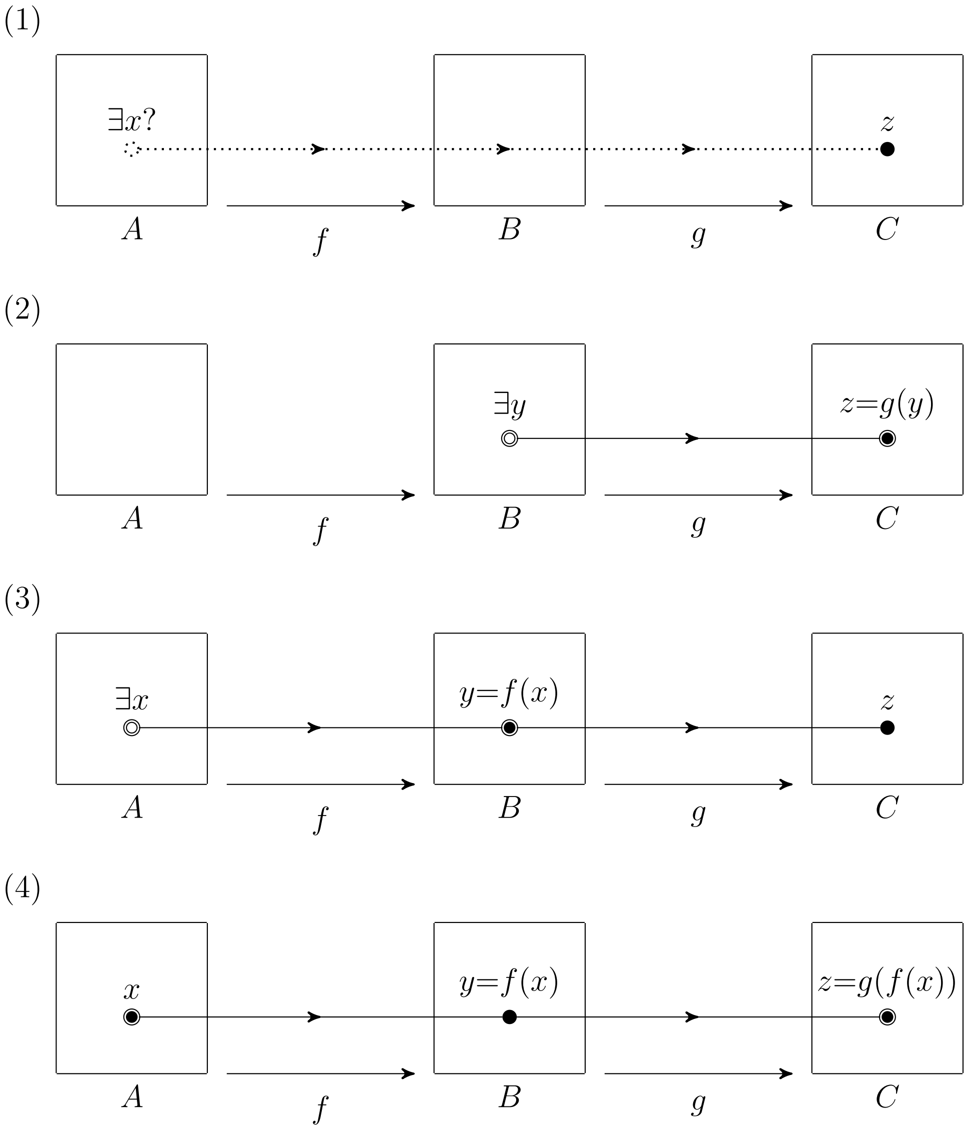
Process of conceiving and naming an appropriate candidate of x which satisfies '$x \in A$' and '$z = (g \circ f)(x)$', with the help of the surjectivity of f and g.

Verification of '$x \in A$' and '$z = (g \circ f)(x)$' for the x named above.

Proof of Statement (1) of Theorem $(\sharp_1)$.

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

Suppose f, g are surjective. [We want to verify that $g \circ f$ is surjective under this assumption.]



Pick any $z \in C$.

[In the rest of the argument, this z is fixed. We want to argue that there is some $x \in A$ satisfying $(g \circ f)(x) = z$.]

For this $z \in C$, by the surjectivity of g, there exists some $y \in B$ such that $g(y) = z$.

[From now on, this y ('generated' by z) is fixed.]

For the same $y \in B$, by the surjectivity of f, there exists some $x \in A$ such that $f(x) = y$.

[From now on, this x ('generated' by y, and hence 'generated' by z ultimately) is fixed.]

For the same $z \in C$, $y \in B$, $x \in A$, we have

$$z = g(y) = g(f(x)) = (g \circ f)(x).$$

It follows that $g \circ f$ is surjective. $\square$

Proof of Statement (1) of Theorem ($\sharp_1$) (without pictures).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

Suppose f, g are surjective.

[We want to verify that $g \circ f$ is surjective under this assumption. By the definition of surjectivity, this is the same as verifying that for any $z \in C$, there exists some $x \in A$ such that $z = (g \circ f)(x)$.]

> Pick any $z \in C$.
> [We want to argue that for this same z, there is some x satisfying $z = (g \circ f)(x)$.]
>
> For this $z \in C$, by the surjectivity of g, there exists some $y \in B$ such that $z = g(y)$.
>
> For the same $y \in B$, by the surjectivity of f, there exists some $x \in A$ such that $y = f(x)$.
>
> For the same $z \in C$, $x \in A$, we have $z = g(f(x)) = (g \circ f)(x)$.

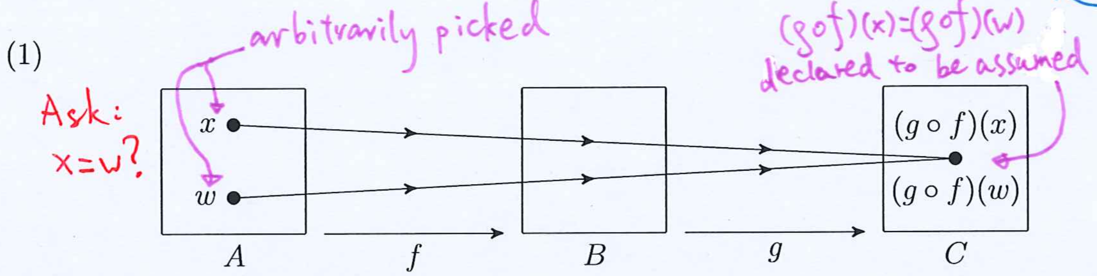
It follows that $g \circ f$ is surjective.

4. **Proof of Statement (2) of Theorem** ($\sharp_1$) .

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

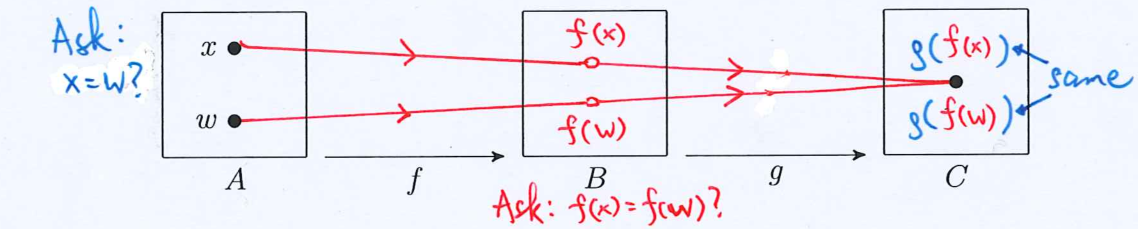
Suppose f, g are injective. [We want to verify that $g \circ f$ is injective under this assumption.]

(1)



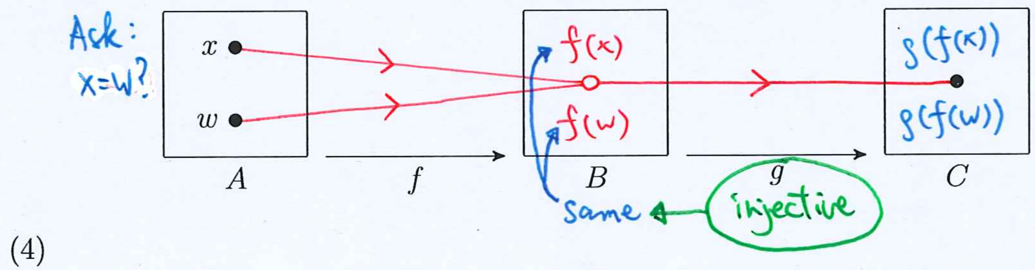
For any $x, w \in A$,
if $(g\circ f)(x)=(g\circ f)(w)$
then $x=w$.

(2)

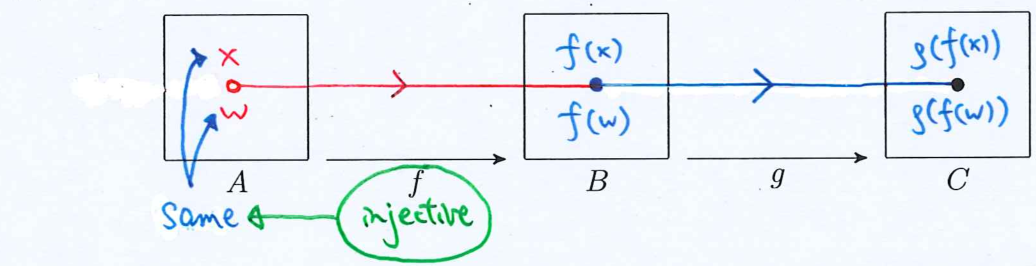


Appropriate argument should read as:
Pick any $x, w \in A$.
Suppose $(g\circ f)(x)=(g\circ f)(w)$.
⋮
[Manipulation leading towards '$x = w$', with the help of the injectivity of f and g.

(3)



(4)

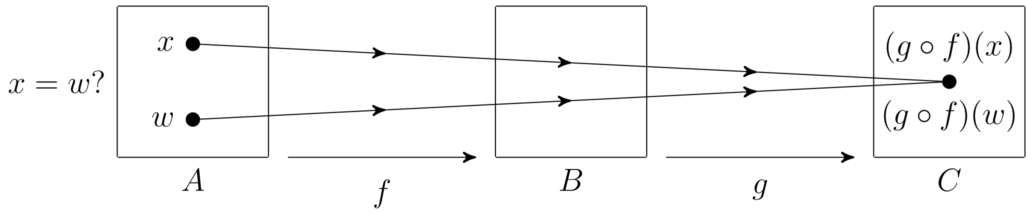


Proof of Statement (2) of Theorem $(\sharp_1)$.

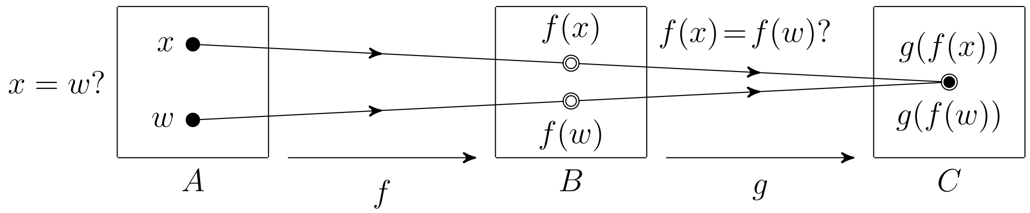
Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

Suppose f, g are injective. [We want to verify that $g \circ f$ is injective under this assumption.]

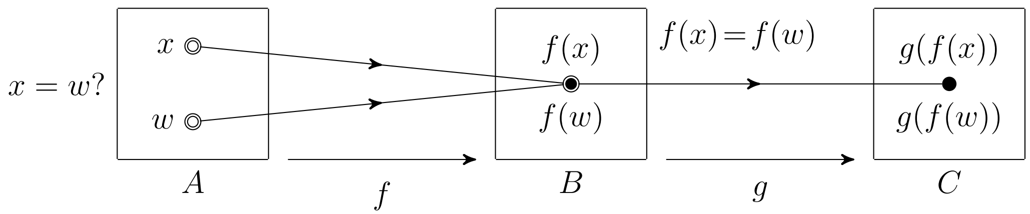
(1)



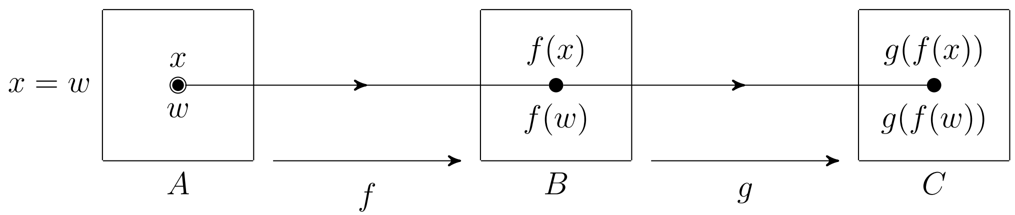
(2)



(3)



(4)



Pick any $x, w \in A$.

[In the rest of the argument, these x, w are fixed.
We want to argue that
if $(g \circ f)(x) = (g \circ f)(w)$ then $x = w$.]

Suppose $(g \circ f)(x) = (g \circ f)(w)$.

[Want to deduce : $x = w$.]

Note that $(g \circ f)(x) = g(f(x))$,
and $(g \circ f)(w) = g(f(w))$.
Then $g(f(x)) = g(f(w))$.

By the injectivity of g,
since $g(f(x)) = g(f(w))$,
we have $f(x) = f(w)$.

By the injectivity of f,
since $f(x) = f(w)$,
we have $x = w$.

It follows that $g \circ f$ is injective. □

Proof of Statement (2) of Theorem ($\sharp_1$) **(without pictures)**.

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

Suppose f, g are injective.

[We want to verify that $g \circ f$ is injective under this assumption. By the definition of injectivity, this is the same as verify that for any $x, w \in A$, if $(g \circ f)(x) = (g \circ f)(w)$ then $x = w$.]

Pick any $x, w \in A$.
[We want to argue that for the same x, w, if $(g \circ f)(x) = (g \circ f(w))$ then $x = w$.]

Suppose $(g \circ f)(x) = (g \circ f)(w)$. Then $g(f(x)) = g(f(w))$.

By the injectivity of g, since $g(f(x)) = g(f(w))$, we have $f(x) = f(w)$.

By the injectivity of f, since $f(x) = f(w)$, we have $x = w$.

It follows that $g \circ f$ is injective.

5. **Theorem** ($\sharp_2$).

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

The following statements hold:

(1) *Suppose $g \circ f$ is surjective. Then g is surjective.*

(2) *Suppose $g \circ f$ is injective. Then f is injective.*

Proof of Theorem ($\sharp_2$). Exercise.

Remark.

The statements below are false. Dis-prove each of them by giving a counter-example.

(1) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.*
Suppose $g \circ f$ is surjective. Then f is surjective.

(2) *Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.*
Suppose $g \circ f$ is injective. Then g is injective.

Further remark.

Which of the statements below are true? Which are false?

(1) *Let* A, B *be sets, and* $f : A \longrightarrow B$, $g : B \longrightarrow A$ *be functions. Suppose* $g \circ f$ *is surjective. Then* $f \circ g$ *is surjective.*

(2) *Let* A, B *be sets, and* $f : A \longrightarrow B$, $g : B \longrightarrow A$ *be functions. Suppose* $g \circ f$ *is injective. Then* $f \circ g$ *is injective.*

(3) *Let* A *be a set, and* $f, g : A \longrightarrow A$ *be functions. Suppose* $g \circ f$ *is surjective. Then* $f \circ g$ *is surjective.*

(4) *Let* A *be a set, and* $f, g : A \longrightarrow A$ *be functions. Suppose* $g \circ f$ *is injective. Then* $f \circ g$ *is injective.*