1. **Example (1).**

Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be the function defined by $f(z) = z^2$ for any $z \in \mathbb{C}$.

Is f surjective? Yes. Justification:

* [What to verify? For any $\zeta \in \mathbb{C}$, there exists some $z \in \mathbb{C}$ such that $f(z) = \zeta$.]

  Pick any $\zeta \in \mathbb{C}$. Note that $\zeta = 0$ or $\zeta \neq 0$.

(†) Suppose $\zeta = 0$. We have $0 \in \mathbb{C}$ and $f(0) = 0 = \zeta$.

(‡) Suppose $\zeta \neq 0$.

[Try to name some appropriate $z \in \mathbb{C}$ satisfying $\overbrace{f(z) = \zeta}^{z^2 = \zeta}$. Roughwork?]

There exists some $\theta \in \mathbb{R}$ such that

$$\zeta = |\zeta| \cdot (\cos(\theta) + i\sin(\theta)).$$

Take $z = \sqrt{|\zeta|} \cdot \left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$.

- By definition, $z \in \mathbb{C}$.
- Also, $f(z) = z^2 = \left[\sqrt{|\zeta|} \cdot \left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)\right]^2$
  $= |\zeta| \cdot (\cos(\theta) + i\sin(\theta)) = \zeta$.

[Roughwork.

Solve the equation $z^2 = \zeta$ with unknown z in $\mathbb{C}$.

Write $\zeta = |\zeta| \cdot (\cos(\theta) + i\sin(\theta))$.

$z^2 = \zeta$

$z = \sqrt{|\zeta|} \cdot \left(\cos\left(\frac{\theta}{2}\right) + i\sin\left(\frac{\theta}{2}\right)\right)$ or...]

It follows that f is surjective.

Remark. Contrast the above result with this statement:

The function $p : \mathbb{R} \longrightarrow \mathbb{R}$ given by $p(x) = x^2$ for any $x \in \mathbb{R}$ is not surjective.

2. **Example (2).**

Let $g : \mathbb{C} \longrightarrow \mathbb{C}$ *be the function defined by* $g(z) = z^3$ *for any* $z \in \mathbb{C}$.

Is g *injective?* No. Justification:

$*$ [What to verify? There exists some $z, w \in \mathbb{C}$ such that $z \neq w$ and $g(z) = g(w)$.]

[Try to name some appropriate distinct $z, w \in \mathbb{C}$ satisfying $\underbrace{g(z) = g(w)}_{z^3 = w^3}$. Roughwork?]

Take $z = 1$, $w = \cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)$.

- $z, w \in \mathbb{C}$
- $z \neq w$.
- $\begin{cases} g(z) = z^3 = 1^3 = 1. \\ g(w) = w^3 = \cdots = 1. \end{cases}$ Then $g(z) = g(w)$.

It follows that g is not injective.

[**Roughwork.**

Ask: What happens when $g(z) = g(w)$?

$g(z) = g(w)$

$\Rightarrow z^3 = w^3$

$\Rightarrow |z|^3 = |w|^3$

$\Rightarrow |z| = |w|$.

Now ask: Can we name some distinct $z, w \in \mathbb{C}$ satisfying $|z| = |w|$ and $g(z) = g(w)$?]

Remark. Contrast the above result with this statement:

The function $q : \mathbb{R} \longrightarrow \mathbb{R}$ *given by* $q(x) = x^3$ *for any* $x \in \mathbb{R}$ *is injective.*

3. **Example (3).**

Let $n \in \mathbb{N}\backslash\{0, 1\}$, *and* $h : \mathbb{C} \longrightarrow \mathbb{C}$ *be the function defined by* $h(z) = z^n$ *for any* $z \in \mathbb{C}$.

Is h *surjective? Is* h *injective?*

4. **Example (4).**

Let $a, b \in \mathbb{C}$. Suppose $a \neq 0$. Define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(z) = az + b$ for any $z \in \mathbb{C}$.

Is f surjective? Yes. Justification:

* [What to verify? For any $\zeta \in \mathbb{C}$, there exists some $z \in \mathbb{C}$ such that $f(z) = \zeta$.]

  Pick any $\zeta \in \mathbb{C}$.

  [Try to name some appropriate $z \in \mathbb{C}$ satisfying $\overbrace{f(z) = \zeta}^{az+b=\zeta}$. Roughwork?]

Take $z = \dfrac{\zeta - b}{a}$.

By definition, $z \in \mathbb{C}$.

Also, $f(z) = az + b = a \cdot \dfrac{\zeta - b}{a} + b = \zeta$.

It follows that f is surjective.

[Roughwork.

Solve the equation $az + b = \zeta$ with unknown z in $\mathbb{C}$.

$$\begin{aligned} az + b &= \zeta \\ az &= \zeta - b \\ z &= \frac{\zeta - b}{a} \end{aligned}$$
]

Is f injective? Yes. Justification:

* [What to verify? For any $z, w \in \mathbb{C}$, if $f(z) = f(w)$ then $z = w$.]

  Pick any $z, w \in \mathbb{C}$. Suppose $f(z) = f(w)$. [Try to deduce $z = w$.]

Then $az + b = aw + b$.

Therefore $az = aw$.

Hence $z = w$.

It follows that f is injective.

5. **Example (5).**

Let $a, b, c \in \mathbb{C}$. *Suppose* $a \neq 0$. *Define the function* $f : \mathbb{C} \longrightarrow \mathbb{C}$ *by* $f(z) = az^2 + bz + c$ *for any* $z \in \mathbb{C}$.

Write $\gamma = -\frac{b}{2a}$, $\Delta = b^2 - 4ac$. Note that $f(z) = a(z-\gamma)^2 - \frac{\Delta}{4a}$ for any $z \in \mathbb{C}$.

Is f surjective? Yes. Justification:

* [What to verify? For any $\zeta \in \mathbb{C}$, there exists some $z \in \mathbb{C}$ such that $f(z) = \zeta$.]

Pick any $\zeta \in \mathbb{C}$.

[Try to name some appropriate $z \in \mathbb{C}$ satisfying $\overbrace{f(z) = \zeta}^{a(z-\gamma)^2 - \frac{\Delta}{4a} = \zeta}$. Roughwork?]

[Roughwork. Solve the quadratic equation $a(z-\gamma)^2 = \zeta + \frac{\Delta}{4a}$ with unknown z in $\mathbb{C}$.
Easy case: '$\zeta = -\frac{\Delta}{4a}$'. Less easy case: '$\zeta \neq -\frac{\Delta}{4a}$'.]

(†) Suppose $\zeta = -\frac{\Delta}{4a}$. Take $z = \gamma$. $\cdots\cdots$ $f(z) = \cdots = \zeta$.

(‡) Suppose $\zeta \neq -\frac{\Delta}{4a}$. Define $\alpha = \frac{1}{a}\left(\zeta + \frac{\Delta}{4a}\right)$. By definition, $\alpha \in \mathbb{C}\backslash\{0\}$.

There exists some $\theta \in \mathbb{R}$ such that $\alpha = |\alpha|(\cos(\theta) + i\sin(\theta))$.

Take $z = \gamma + \sqrt{|\alpha|} \cdot \left(\cos(\frac{\theta}{2}) + i\sin(\frac{\theta}{2})\right)$. $\cdots\cdots$ $f(z) = \cdots = \zeta$.

It follows that f is surjective.

Example (5).

Let $a, b, c \in \mathbb{C}$. Suppose $a \neq 0$. Define the function $f : \mathbb{C} \longrightarrow \mathbb{C}$ by $f(z) = az^2 + bz + c$ for any $z \in \mathbb{C}$.

Write $\gamma = -\dfrac{b}{2a}$, $\Delta = b^2 - 4ac$. Note that $f(z) = a(z-\gamma)^2 - \dfrac{\Delta}{4a}$ for any $z \in \mathbb{C}$.

Is f injective?

No. Justification?

$*$ [What to verify? There exist some $z, w \in \mathbb{C}$ such that $z \neq w$ and $f(z) = f(w)$.]

[Try to name some appropriate distinct $z, w \in \mathbb{C}$ satisfying $f(z) = f(w)$. Roughwork?]

[Roughwork.

Ask: what happens when $f(z) = f(w)$?

Now ask: Can we name some distinct $z, w \in \mathbb{C}$ satisfying $|z-\gamma| = |w-\gamma|$ and $f(z) = f(w)$?

$f(z) = f(w)$

$\Rightarrow a(z-\gamma)^2 - \dfrac{\Delta}{4a} = a(w-\gamma)^2 - \dfrac{\Delta}{4a}$

$\Rightarrow (z-\gamma)^2 = (w-\gamma)^2$

$\Rightarrow |z-\gamma|^2 = |w-\gamma|^2$

$\Rightarrow |z-\gamma| = |w-\gamma|$]

Take $z = \gamma + 1$, $w = \gamma - 1$.

Note that $z, w \in \mathbb{C}$ and $z \neq w$.

$$f(z) = a - \frac{\Delta}{4a} = f(w).$$

It follows that f is not injective.

Known by now:

- Every 'linear function from $\mathbb{C}$ to $\mathbb{C}$' is both surjective and injective.
- Every 'quadratic function from $\mathbb{C}$ to $\mathbb{C}$' is surjective and not injective.

Question.

How about cubic functions from $\mathbb{C}$ to $\mathbb{C}$?

Answer.

- Let $a, b, c, d \in \mathbb{C}$. Suppose $a \neq 0$. Define the function $f : \mathbb{C} \to \mathbb{C}$ by $f(z) = az^3 + bz^2 + cz + d$ for any $z \in \mathbb{C}$.

  Then f is surjective and not injective.

Why? This is a consequence of the result below and the Factor Theorem.

Cardano-and-Tartaglia Theorem on cubic equations:

- Let A, B, C, D be complex numbers. Suppose $A \neq 0$.

  Then the equation $Az^3 + Bz^2 + Cz + D = 0$ with unknown z in $\mathbb{C}$ has at least one solution in $\mathbb{C}$, given by the $\underbrace{\text{'cubic formula'} \cdots\cdots}_{\text{[Find out what it is by yourself.]}}$'.

6. **Polynomial functions on $\mathbb{C}$.**

We introduce these definitions:

(a) *Let $n \in \mathbb{N}$. A* **degree-n polynomial with complex coefficients and with indeterminate** *z is an expression of the form $a_n z^n + \cdots + a_1 z + a_0$ in which $a_0, a_1, \cdots, a_n \in \mathbb{C}$ and $a_n \neq 0$.*

(b) *Let $f : \mathbb{C} \longrightarrow \mathbb{C}$ be a function. f is said to be a* **degree-n polynomial function (with complex coefficients) on $\mathbb{C}$** *if there exist some $a_0, a_1, \cdots, a_n \in \mathbb{C}$ such that $a_n \neq 0$ and $f(z) = a_n z^n + \cdots + a_1 z + a_0$ for any $z \in \mathbb{C}$.*

The examples above are special cases of these results:

Theorem (1).

Let $n \in \mathbb{N} \backslash \{0, 1\}$. Every degree-$n$ polynomial function on $\mathbb{C}$ is surjective.

Theorem (2).

Let $n \in \mathbb{N} \backslash \{0, 1\}$. Every degree-$n$ polynomial function on $\mathbb{C}$ is not injective.

Polynomial functions on $\mathbb{C}$.

Theorem (1).

Let $n \in \mathbb{N} \backslash \{0, 1\}$. Every degree-$n$ polynomial function on $\mathbb{C}$ is surjective.

Theorem (2).

Let $n \in \mathbb{N} \backslash \{0, 1\}$. Every degree-$n$ polynomial function on $\mathbb{C}$ is not injective.

Theorem (1) is logically equivalent to the **Fundamental Theorem of Algebra**:

Every non-constant polynomial with complex coefficient has a root in $\mathbb{C}$.

We can deduce Theorem (2) from Theorem (1) with the help of the **Factor Theorem** :

Let $\alpha \in \mathbb{C}$, and $p(z)$ be a degree-n polynomial (with complex coefficients).

Suppose α is a root of $p(z)$.

Then there is a degree-$(n-1)$ polynomial $q(z)$ (with complex coefficients) so that $p(z) = (z - \alpha)q(z)$ as polynomials.