1. Let $f_1, f_2, f_3, f_4, f_5, f_6, f_7, f_8 : \mathbb{R} \longrightarrow \mathbb{R}$ be the functions defined by

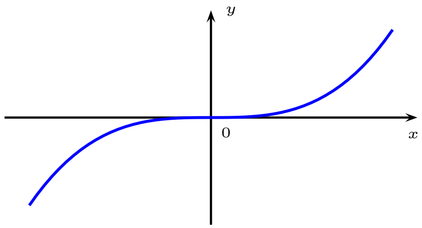
$$f_1(x) = 0.1x^3, \quad f_2(x) = \sqrt[5]{x} - 1, \quad f_3(x) = x^5 - 2x^3 + x, \quad f_4(x) = 0.25x^2 \sin(10x),$$

$$f_5(x) = 1.3^x, \quad f_6(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}, \quad f_7(x) = \frac{1}{x^2+1}, \quad f_8(x) = 4^{\sin(4x)}$$

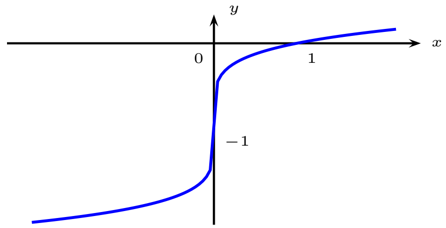
for any $x \in \mathbb{R}$.

Rough sketches of the respective graphs of the above functions:

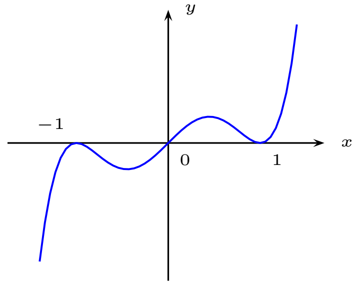
Graph of f_1:



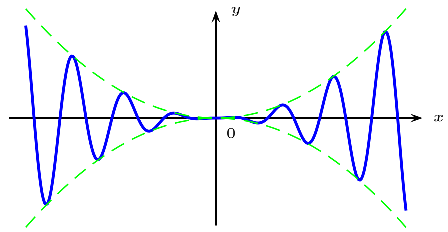
Graph of f_2:



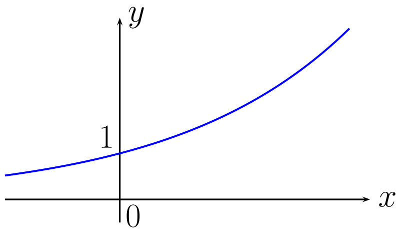
Graph of f_3:



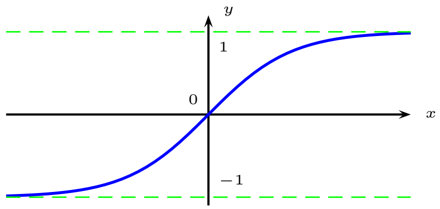
Graph of f_4:



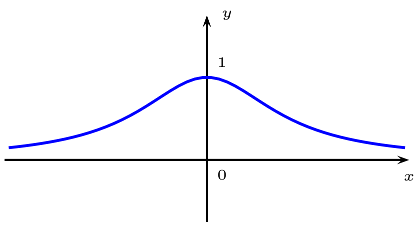
Graph of f_5:



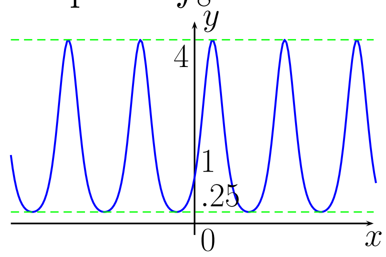
Graph of f_6:



Graph of f_7:



Graph of f_8



2. Which of $f_1, \cdots, f_8$ is/are surjective? Which not?

- f_1, f_2, f_3, f_4 are surjective.
- f_5, f_6, f_7, f_8 are not surjective.

Question. How to see the answer for such functions from $\mathbb{R}$ to $\mathbb{R}$?

Answer (b1). Inspect the graph of $f_1, \cdots, f_8$ on the 'coordinate plane'.

- $i = 1, 2, 3, 4$. Why surjective?
  At each 'altitude' $b \in \mathbb{R}$, the horizontal line $y = b$ cuts the graph of f_i at least once.
  Some $x_b \in \mathbb{R}$ satisfies $y = f_i(x_b)$.
- $i = 5, 6, 7, 8$. Why not surjective?
  At some 'altitude' $b_0 \in \mathbb{R}$, the horizontal line $y = b_0$ cuts the graph of f_i nowhere.
  No $x \in \mathbb{R}$ satisfies $b_0 = f_i(x)$.

Answer (b2). Re-interpret (b1) in terms of solving equations.

- $i = 1, 2, 3, 4$. Why surjective?
  For each $b \in \mathbb{R}$, the equation $b = f_i(u)$ with 'unknown' u in $\mathbb{R}$ has at least one solution in $\mathbb{R}$.
- $i = 5, 6, 7, 8$. Why not surjective?
  There is some $b_0 \in \mathbb{R}$ for which the equation $b_0 = f(u)$ with 'unknown' u in $\mathbb{R}$ has no solution in $\mathbb{R}$.

Answer (a). Directly verify the condition (S) or its negation respectively.

- $i = 1, 2, 3, 4$. Surjectivity? [Recall the statement (S).]
  - ∗ How do we check the surjectivity of f_1, in practice? $\forall y \in \mathbb{R}, \exists x \in \mathbb{R}$ such that $y = f_1(x)$.

    Pick any $y \in \mathbb{R}$. [This y is kept fixed in the discussion below.]
    [We name a candidate $x \in \mathbb{R}$ for which $y = f_1(x)$. An appropriate candidate is given by a solution of the equation $y = f_1(u)$ with unknown u in $\mathbb{R}$.]

    Take $x = \sqrt[3]{10y}$.

    ⊙ By definition, $x \in \mathbb{R}$.

    ⊙ Also, $f_1(x) = 0.1x^3 = 0.1(\sqrt[3]{10y})^3 = (0.1)\cdot(10y) = y$.

    It follows that f_1 is surjective. □

    [Roughwork: Solve $y = f_1(u)$ with unknown u in $\mathbb{R}$.
    $y = 0.1u^3$
    $u^3 = 10y$
    $u = \sqrt[3]{10y}$]

  - ∗ How about f_2? [Exercise.]

    Pick any $y \in \mathbb{R}$.

    Take $x = (y+1)^5$.

    · By definition, $x \in \mathbb{R}$.

    · Also, $f_2(x) = \sqrt[5]{x} - 1 = \sqrt[5]{(y+1)^5} - 1 = (y+1) - 1 = y$.

    It follows that f_2 is surjective. □

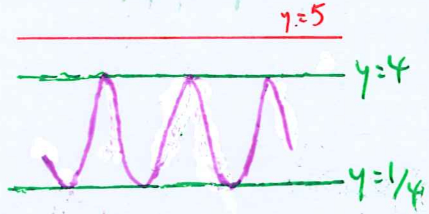
    [Roughwork: Solve $y = f_2(u)$ with unknown u in $\mathbb{R}$.
    $y = \sqrt[5]{u} - 1$
    $\sqrt[5]{u} = y+1$
    $u = (y+1)^5$]

Remark. Things are more difficult in practice for f_3, f_4, when we do not make use of the calculus. (Why?)

Answer (a). Directly verify the condition (S) or its negation respectively.

- $i = 5, 6, 7, 8$. Non-surjectivity? [Recall the statement $\sim(S)$.]
  - How do we check the non-surjectivity of f_8, in practice? $\exists y_0 \in \mathbb{R}$ such that $(\forall x \in \mathbb{R}, y_0 \neq f_8(x))$.

    [Name a candidate $y_0 \in \mathbb{R}$ for which $y_0 \neq f_8(x)$ for any $x \in \mathbb{R}$. We are aware that for any $x \in \mathbb{R}$, $4^{-1} \leq 4^{\sin(4x)} \leq 4$.]



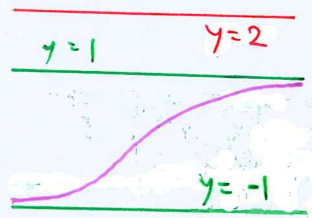
Take $y_0 = 5$.

- Note that $y_0 \in \mathbb{R}$.
- Pick any $x \in \mathbb{R}$. We have $f_8(x) = 4^{\sin(4x)} \leq 4 < 5 = y_0$.

  Hence $y_0 \neq f_8(x)$.

It follows that f_8 is not surjective. $\square$

  - How about f_6?



[Roughwork: Observe that for any $x \in \mathbb{R}$,

$$|f_6(x)| = \left|\frac{e^x - e^{-x}}{e^x + e^{-x}}\right| = \frac{|e^x - e^{-x}|}{e^x + e^{-x}} \leq \frac{|e^x| + |e^{-x}|}{e^x + e^{-x}} = 1.$$]

Take $y_0 = 2$.

- Note that $y_0 \in \mathbb{R}$.
- Pick any $x \in \mathbb{R}$. We have $|f_6(x)| = \left|\frac{e^x - e^{-x}}{e^x + e^{x}}\right| \leq 1 < 2 = y_0$. Then $f_6(x) \neq y_0$.

It follows that f_6 is not surjective. $\square$

  - How about f_5, f_7? [Exercise.]

3. Which of $f_1, \cdots, f_8$ is/are injective? Which not?

- f_1, f_2, f_5, f_6 are injective.
- f_3, f_4, f_7, f_8 are not injective.

Question. How to see which is injective and which not, for such a real-valued function of one real variable?

Answer (b1). Inspect the graph of $f_1, \cdots, f_8$.

- $i = 1, 2, 5, 6$. Why injective?
  At each 'altitude' $b \in \mathbb{R}$, the horizontal line $y = b$ cuts the graph of $f_i$at most once: no two distinct x, w satisfy $f_i(x) = f_i(w)$.
- $i = 3, 4, 7, 8$. Why not injective?
  At some 'altitude' $b_0 \in \mathbb{R}$, the horizontal line $y = b_0$ cuts the graph of f_i twice or more: some distinct x, w satisfy $f_i(x) = f_i(w)$.

Answer (b2). We re-interpret (b1) in terms of solving equations.

- $i = 1, 2, 5, 6$. Why injective?
  For each $b \in \mathbb{R}$, the equation $b = f_i(u)$ with 'unknown' u in $\mathbb{R}$ has at most one solution in $\mathbb{R}$.
- $i = 3, 4, 7, 8$. Why not injective?
  There is some value $b_0 \in \mathbb{R}$ for which the equation $b_0 = f_i(u)$ with 'unknown' u in $\mathbb{R}$ has two or more solutions in $\mathbb{R}$.

Answer (a). Directly verify the condition (I) or its negation respectively.

- $i = 1, 2, 5, 6$. Injectivity? [Recall the statement (I).]

  * How do we check the injectivity of f_6, in practice? $\forall x, w \in \mathbb{R}$, if $f_6(x) = f_6(w)$ then $x = w$.

    Pick any $x, w \in \mathrm{IR}$. [These x, w are fixed in the discussion below. We verify that if $f_6(x) = f_6(w)$ then $x = w$.]

    Suppose $f_6(x) = f_6(w)$.

    Then $\dfrac{e^x - e^{-x}}{e^x + e^{-x}} = \dfrac{e^w - e^{-w}}{e^w + e^{-w}}$.

    Therefore $e^{x+w} + e^{x-w} - e^{-x+w} - e^{-x-w} = (e^x - e^{-x})(e^w + e^{-w}) = (e^x + e^{-x})(e^w - e^{-w})$
    $= e^{x+w} - e^{x-w} + e^{-x+w} - e^{-x-w}$.

    Hence $2e^{x-w} = 2e^{-x+w}$. Then $x - w = -x + w$. We have $x = w$.

    It follows that f_6 is injective. □

  * How about f_1?

    Pick any $x, w \in \mathbb{R}$. Suppose $f_1(x) = f_1(w)$.

    Then $0.1x^3 = 0.1w^3$.

    Therefore $(x - w)(x^2 + xw + w^2) = x^3 - w^3 = 0$.

    $\vdots \leftarrow$ [your work.]

    Hence $x = w$.

    It follows that f_1 is injective. □

  * How about f_2, f_5? [Exercise.]

Answer (a). Directly verify the condition (I) or its negation respectively.

- $i = 3, 4, 7, 8$. Non-injectivity? [Recall the statement $\sim(I)$.]

  How do we check the non-injectivity of f_7, in practice? $\exists\, x_0, w_0 \in \mathbb{R}$ such that $x_0 \neq w_0$ and $f_7(x_0) = f_7(w_0)$.

  [Name $x_0, w_0 \in \mathbb{R}$ for which $x_0 \neq w_0$ and $f_7(x_0) = f_7(w_0)$. Try this roughwork: Start with the 'relation' '$f_7(x_0) = f_7(w_0)$' to see what may prevent us from obtaining '$x_0 = w_0$'.]

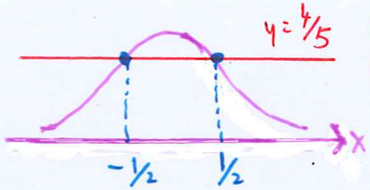
Take $x_0 = \frac{1}{2}$, $w_0 = -\frac{1}{2}$.

- $x_0, w_0 \in \mathbb{R}$.
- $x_0 \neq w_0$.
- $f_7(x_0) = \dfrac{1}{1+x_0^2} = \dfrac{1}{1+(\frac{1}{2})^2} = \dfrac{4}{5}$.

  $f_7(w_0) = \dfrac{1}{1+w_0^2} = \dfrac{1}{1+(-\frac{1}{2})^2} = \dfrac{4}{5}$.

  So $f_7(x_0) = f_7(w_0)$.

  It follows that f_7 is not injective. $\square$



[Roughwork:

Ask: What happens when $f_7(x_0) = f_7(w_0)$?

$$f_7(x_0) = f_7(w_0)$$
$$\Rightarrow \frac{1}{1+x_0^2} = \frac{1}{1+w_0^2}$$
$$\Rightarrow 1+x_0^2 = 1+w_0^2$$
$$\Rightarrow x_0^2 = w_0^2$$
$$\Rightarrow |x_0| = |w_0|.$$

Now ask: Can we name some distinct $x_0, w_0 \in \mathbb{R}$ satisfying $|x_0| = |w_0|$ and $f_7(x_0) = f(w_0)$?]

How about f_3, f_4, f_8? [Exercise.]