1. **Definition of surjectivity**.

   Let A, B be sets, and $f : A \longrightarrow B$ be a function from A to B. f is said to be **surjective** *if the statement (S) holds:*

   (S): For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.

   In the symbols of logic, (S) is
   $$(\forall y \in B)[(\exists x \in A)(y = f(x))].$$
   So for each $y \in B$, (S) gives an existence statement.

2. **Pictorial visualizations of surjective functions**.

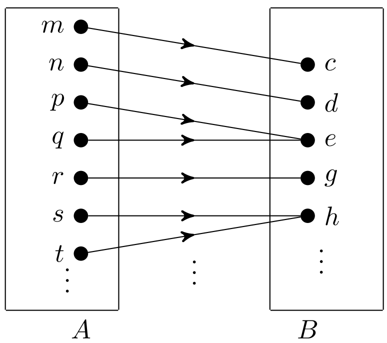
   (a) 'Blobs-and-arrows diagram'.
   f is surjective exactly when every element of B is 'pointed' at by at least one element of A.

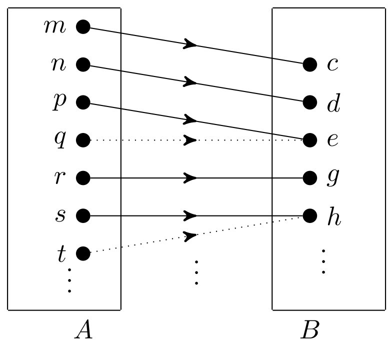
   (b) 'Coordinate plane diagram'.
   f is surjective exactly when, for each $b \in B$, the equation $b = f(u)$ with 'unknown' u has at least one solution in A.

3. **Example of a surjective function and its pictorial visualizations**.

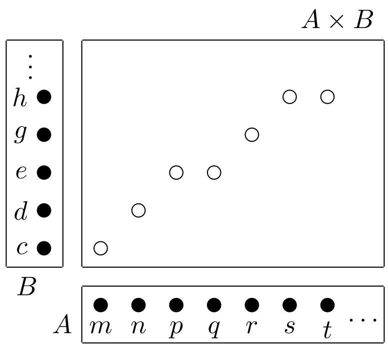
   Let $A = \{m, n, p, q, r, s, t, \cdots\}$, $B = \{c, d, e, g, h \cdots\}$, and $f : A \longrightarrow B$ be the surjective function defined by $f(m) = c$, $f(n) = d$, $f(p) = f(q) = e$, $f(r) = g$, $f(s) = f(t) = h$, ...

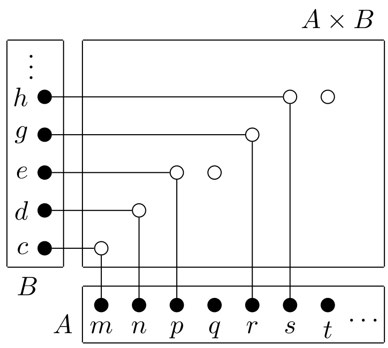
   (a) *'Blobs-and-arrows diagram'. Every element of B is 'pointed' at by at least one element of A.*





   (b) *'Coordinate plane diagram'. For each $b \in B$, the equation $b = f(u)$ with 'unknown' u has at least one solution in A.*





4. **How to check that a given function $f : A \longrightarrow B$ is surjective?**

   What to check?

   (S): For any $y \in B$, there exists some $x \in A$ such that $y = f(x)$.

   Procedure:

   (1) Pick any $y \in B$. (From this moment on this y is fixed.)

   (2a) Name an appropriate x_y which you believe will satisfy both $x_y \in A$ and $y = f(x_y)$, when it is immediately clear which x_y is to be named.

   (2b) When it is not immediately clear which x_y is to be named, do some roughwork:

   * With y fixed, solve the equation $y = f(u)$ with 'unknown' u in A.
     One such solution is an appropriate candidate x_y to be named.

   (3) Verify that the x_y just named indeed satisfy $x_y \in A$ and $y = f(x_y)$.

5. **Negation of surjectivity**.

Let A, B be sets, and $f : A \longrightarrow B$ be a function. $f : A \longrightarrow B$ is not surjective iff the statement $\sim(S)$ holds:

> $\sim(S)$: *There exists some $y_0 \in B$ such that for any $x \in A$, $y_0 \neq f(x)$.*

In the symbols of logic, $\sim(S)$ is

$$(\exists y_0 \in B)[(\forall x \in A)(y_0 \neq f(x))].$$

6. **Pictorial visualizations of non-surjective functions**.

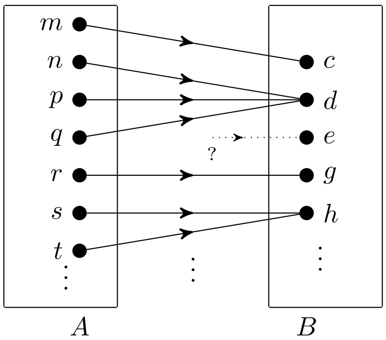
   (a) 'Blobs-and-arrows diagram'.
   f is not surjective exactly when some element of B is 'pointed' at by no element of A.

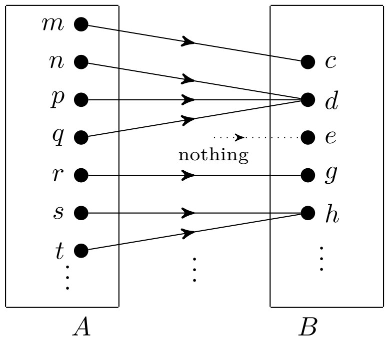
   (b) 'Coordinate plane diagram'.
   f is not surjective exactly when there is some element b_0 of B for which the equation $b_0 = f(u)$ with 'unknown' u has no solution in A.

7. **Example of a non-surjective functions and its pictorial visualizations**.

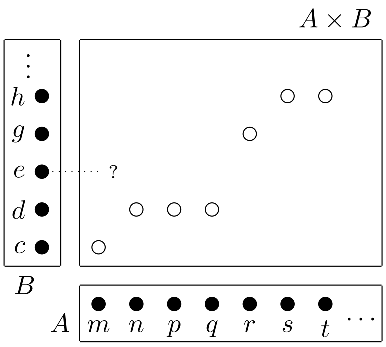
Let $A = \{m, n, p, q, r, s, t, \cdots\}$, $B = \{c, d, e, g, h \cdots\}$, and $f : A \longrightarrow B$ be the non-surjective function defined by $f(m) = c$, $f(n) = f(p) = f(q) = d$, $f(r) = g$, $f(s) = f(t) = h$, ...

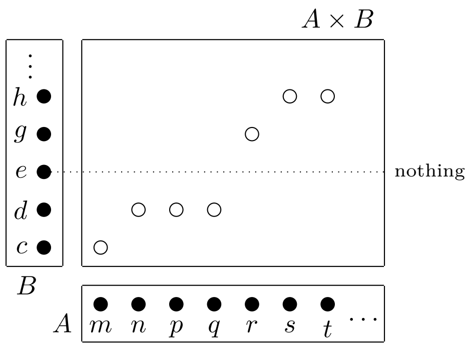
   (a) *'Blobs-and-arrows diagram'. Some element of B is 'pointed' at by no element of A.*





   (b) *'Coordinate plane diagram'. There is some element b_0 of B for which the equation $b_0 = f(u)$ with 'unknown' u has no solution in A.*





8. **How to check that a given function $f : A \longrightarrow B$ is not surjective?**

What to check?

> $\sim(S)$: *There exists some $y_0 \in B$ such that for any $x \in A$, $y_0 \neq f(x)$.*

Procedure:

(1a) Name appropriate 'concrete' $y_0 \in B$ which you believe will satisfy $y_0 \neq f(x)$ for any $x \in A$, when it is immediately clear which y_0 to be named.

(1b) When it is not immediately clear which y_0 is to be named, do some roughwork:

   (1b.i) First look for a necessary condition in terms of the 'value' of b for the equation $b = f(u)$ with 'unknown' u to have a solution, by asking this question:

   'What can be said of b if $b = f(u)$ has a solution?'

   (1b.ii) Look for an appropriate y_0 which has to be something failing to satisfy this necessary condition.

(2) Then go to (†) or (‡):

   (†) Verify that for any $x \in A$, $f(x) \neq y_0$.

   (‡) Obtain a contradiction under the assumption 'there existed some $x_0 \in A$ such that $f(x_0) = y_0$'.

9. **Definition of injectivity**.
Let A, B be sets, and $f : A \longrightarrow B$ be a function from A to B. f is said to be **injective** *if the statement (I) holds:*

  (I): For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.

Various re-formulation of (I):

  (I'): For any $x, w \in A$, if $x \neq w$ then $f(x) \neq f(w)$.
  (I''): For any $y \in B$, for any $x, w \in A$, if $(y = f(x)$ and $y = f(w))$ then $x = w$.

In the symbols of logic, (I), (I'), (I'') are:

$$(I): \quad (\forall x \in A)(\forall w \in A)[(f(x) = f(w)) \rightarrow (w = x)].$$
$$(I'): \quad (\forall x \in A)(\forall w \in A)[(x \neq w) \rightarrow (f(x) \neq f(w))].$$
$$(I''): \quad (\forall y \in B)(\forall x \in A)(\forall w \in A)[[(y = f(x)) \wedge (y = f(w))] \rightarrow (x = w)].$$

So for each $y \in B$, (I'') gives a uniqueness statement.

10. **Negation of injectivity**.
Let A, B be sets, and $f : A \longrightarrow B$ be a function. f is not injective iff the statement $\sim(I)$ holds:

  $\sim(I)$: There exist some $x_0, w_0 \in A$ such that $f(x_0) = f(w_0)$ and $x_0 \neq w_0$.

Re-formulation of $\sim(I)$ as $\sim(I'')$:

  $\sim(I'')$: There exist some $y_0 \in B$, $x_0, w_0 \in A$ such that $(y_0 = f(x_0)$ and $y_0 = f(w_0)$ and $x_0 \neq w_0)$.

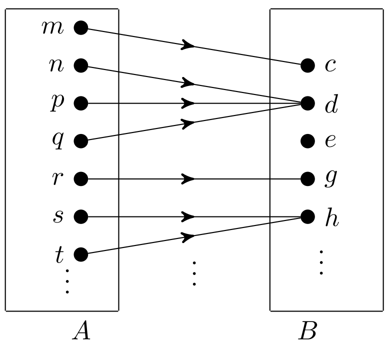
$\sim(I)$, $\sim(I')$ are exactly the same as each other. (Why?)
In the symbols of logic, $\sim(I)$, $\sim(I')$, $\sim(I'')$ are:

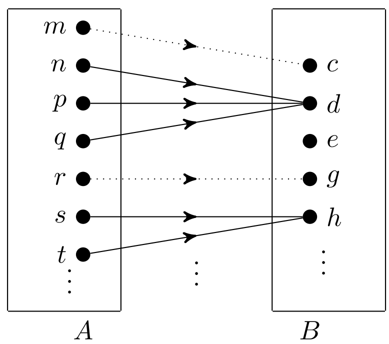
$$\sim(I), \sim(I'): \quad (\exists x_0 \in A)(\exists w_0 \in A)[(f(x_0) = f(w_0)) \wedge (x_0 \neq w_0)].$$
$$\sim(I''): \quad (\exists y_0 \in B)(\exists x_0 \in A)(\exists w_0 \in A)[(y_0 = f(x_0)) \wedge (y_0 = f(w_0)) \wedge (x_0 \neq w_0)].$$

11. **Pictorial visualizations of non-injective functions**.
  (a) 'Blobs-and-arrows diagram'.
  f is not injective exactly when some element of B is 'pointed' at by two or more elements of A.
  (b) 'Coordinate plane diagram'.
  f is not injective exactly when there is some element b_0 of B for which the equation $b_0 = f(u)$ with 'unknown' u has two or more solutions in A.

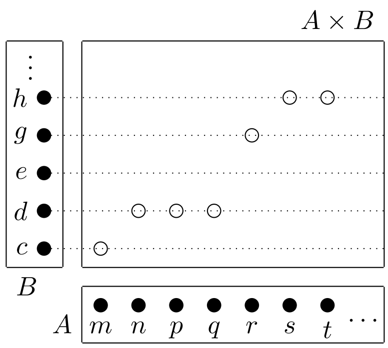
12. **Example of a non-injective function and its pictorial visualizations**.
Let $A = \{m, n, p, q, r, s, t, \cdots\}$, $B = \{c, d, e, g, h \cdots\}$, and $f : A \longrightarrow B$ be the non-injective function defined by $f(m) = c$, $f(n) = f(p) = f(q) = d$, $f(r) = g$, $f(s) = f(t) = h$, ...

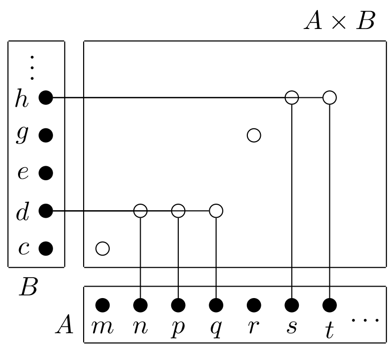
  (a) *'Blobs-and-arrows diagram'. Some element of B is 'pointed' at by two or more elements of A.*





  (b) *'Coordinate plane diagram'. There is some element b_0 of B for which the equation $b_0 = f(u)$ with 'unknown' u has two or more solutions in A.*





13. **Pictorial visualizations of injective functions**.

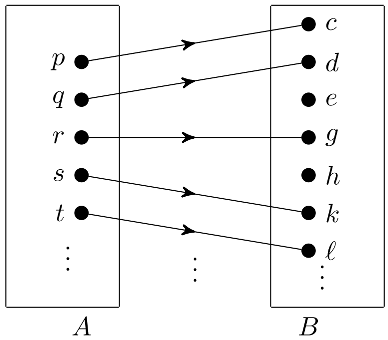
    (a) 'Blobs-and-arrows diagram'.
    f is injective exactly when every element of B is 'pointed' at by at most one element of A.

    (b) 'Coordinate plane diagram'.
    f is injective exactly when, for each $b \in B$, the equation $b = f(u)$ with 'unknown' u has at most one solution in A.

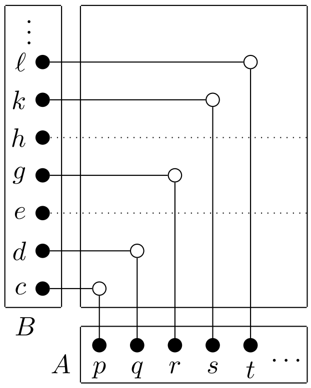
14. **Example of an injective function and its pictorial visualization**.
    Let $A = \{p, q, r, s, t, \cdots\}$, $B = \{c, d, e, g, h, k, \ell \cdots\}$, and $f : A \longrightarrow B$ be the injective function defined by $f(p) = c$, $f(q) = d$, $f(r) = g$, $f(s) = k$, $f(t) = \ell$, ...

    (a) *Blobs-and-arrows diagram'.*

    Every element of B is 'pointed' at by at most one element of A.



    (b) *'Coordinate plane diagram'.*
    For each $y \in B$, the equation $y = f(x)$ with 'unknown' x has at most one solution in A.



15. **How to check that a given function $f : A \longrightarrow B$ is injective?**
    What to check?

    (I): *For any $x, w \in A$, if $f(x) = f(w)$ then $x = w$.*

    Procedure:

    (1) Pick any $x, w \in A$. (From this moment on, x, w are fixed.)
    (2) Suppose $f(x) = f(w)$. Then verify that $x = w$.

16. **How to check that a given function $f : A \longrightarrow B$ is not injective?**
    What to check?

    $\sim(I)$: *There exists some $x_0, w_0 \in A$ such that $f(x_0) = f(w_0)$ and $x_0 \neq w_0$.*

    Procedure:

    (1a) Name appropriate 'concrete' distinct $x_0, w_0 \in A$ which you believe will satisfy $f(x_0) = f(w_0)$, when it is immediately clear which x_0, w_0 are to be named.
    (1b) When it is not immediately clear which distinct x_0, w_0 are to be named, do some roughwork:
      * Look for an appropriate $b_0 \in B$ for which the equation $b_0 = f(u)$ with 'unknown' u has at least two solutions in A. Two such solutions are to be named as x_0, w_0.
    (2) Verify that $f(x_0) = f(w_0)$ indeed.