1. **Real-valued functions of one real variable in school mathematics**.

Below is a typical 'explanation' of the notion of real valued functions of one real variable in school mathematics:

Let D be a subset of $\mathbb{R}$ (very often $\mathbb{R}$ itself or $\mathbb{R}$ with a few points deleted).

A real-valued function defined on D is a 'rule of assignment' from D to $\mathbb{R}$, so that

each number in D is being assigned to exactly one element of $\mathbb{R}$.

When we refer to such a function by f, the set D will be referred to as the domain of this function f.

Whenever $x \in D$, $y \in \mathbb{R}$ and x is assigned to y, we write $y = f(x)$.

The set $G = \{(x, f(x)) \mid x \in D\}$ is called the graph of f. Note that $G \subset \mathbb{R}^2$.

How about 'general' functions?

~~Below is a typical 'explanation' of the notion of real valued functions of one real variable in school mathematics:~~

Let A, B be sets.

~~Let D be a subset of $\mathbb{R}$ (very often $\mathbb{R}$ itself or $\mathbb{R}$ with a few points deleted).~~

A ~~real valued~~ *function* from A to B ~~*defined on D*~~ *is a 'rule of assignment' from* $\overset{A}{\cancel{D}}$ *to* $\overset{B}{\cancel{\mathbb{R}}}$*, so that* each element of A ~~*each number in D*~~ *is being assigned to exactly one element of* $\overset{B}{\cancel{\mathbb{R}}}$*.*

When we refer to such a function by f, the set $\overset{A}{\cancel{D}}$ *will be referred to as the domain of this function f.* The set B is called the range of f.

Whenever $x \in \overset{A}{\cancel{D}}$, $y \in \overset{B}{\cancel{\mathbb{R}}}$ *and x is assigned to y, we write $y = f(x)$.*

The set $G = \{\underset{???}{(x, f(x))} \mid x \in \underset{A?}{D}\}$ *is called* $\underset{???}{\text{the graph of } f}$*. Note that* $G \subset \underset{???}{\mathbb{R}^2}$*.*

2. **In-formal definition of function**.

Let A, B be sets.

A function from A to B is a 'rule of assignment' from A to B, so that

each element of A is being assigned to exactly one element of B.

Conventions and notations.

- When we denote such a function by f, we refer to it as $f : A \longrightarrow B$.
  Whenever $x \in A$, $y \in B$ and x is assigned to y, we write $y = f(x)$ (or $x \underset{f}{\mapsto} y$).
- A is called the **domain** of f. B is called the **range** of f.

Remark. We postpone the generalization of the notion of graphs of functions.

3. **'Blobs-and-arrows diagrams'.**

We may visualize a function by its '**blobs-and-arrows diagram**'.

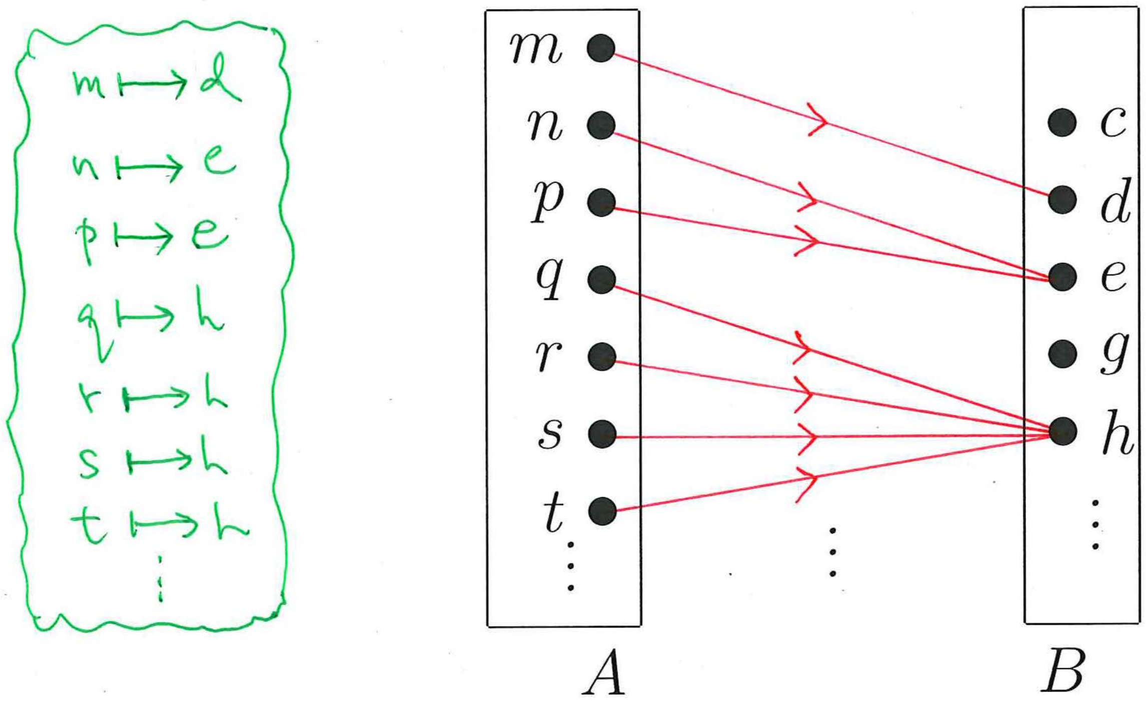
We illustrate the idea with the example below:

Let $A = \{m, n, p, q, r, s, t, ...\}$, $B = \{c, d, e, g, h, ...\}$, *and* $f : A \longrightarrow B$ *be defined by*

$$f(m) = d, f(n) = e, f(p) = e, f(q) = h, f(r) = h, f(s) = h, f(t) = h, \cdots .$$

By definition, f *assigns* m *to* d, n *to* e, p *to* e, q *to* h, r *to* h, s *to* h, t *to* h, $\cdots$.

We draw the 'blobs-and-arrows diagram' of the function f *as:*



4. **Notion of equality for functions.**

We regard two functions to be the same as each other exactly when they 'determine the same assignment'.

Definition.

Let A_1, A_2, B_1, B_2 be sets, and $f_1 : A_1 \longrightarrow B_1$, $f_2 : A_2 \longrightarrow B_2$ be functions.

We agree to say that f_1 is **equal** *to f_2 as functions, and to write $f_1 = f_2$, exactly when*

$$A_1 = A_2 \text{ and } B_1 = B_2 \text{ and } f_1(x) = f_2(x) \text{ for any } x \in A_1.$$

Examples and non-examples.

① $f_1 : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by
$f_1(x) = x+1$ for any $x \in \mathbb{R} \setminus \{1\}$;
$f_2 : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by
$f_2(x) = \dfrac{x^2-1}{x-1}$ for any $x \in \mathbb{R} \setminus \{1\}$.

- Are f_1, f_2 equal to each other?
- Yes.

② $g_1 : \mathbb{R} \to \mathbb{R}$ is given by
$g_1(x) = x+1$ for any $x \in \mathbb{R}$;
$g_2 : \mathbb{R} \setminus \{1\} \to \mathbb{R}$ is given by
$g_2(x) = \dfrac{x^2-1}{x-1}$ for any $x \in \mathbb{R} \setminus \{1\}$.

- Are g_1, g_2 equal to each other?
- No. (Reason: Domains do not agree.)

$g_3 : \mathbb{R} \to \mathbb{C}$ is given by
$g_3(x) = x+1$ for any $x \in \mathbb{R}$;

- Are g_1, g_3 equal to each other?
- No. (Reason: Ranges do not agree.)

③ $h_1 : \mathbb{R} \to \mathbb{R}$ is given by
$h_1(x) = \begin{cases} -1 & \text{if } x \leq 0 \\ 1 & \text{if } x > 0; \end{cases}$
$h_2 : \mathbb{R} \to \mathbb{R}$ is given by
$h_2(x) = \begin{cases} -1 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0. \end{cases}$

- Are h_1, h_2 equal to each other?
- No. (Reason: 'Formulae' do not agree, for instance, at 0.)

5. **Compositions.**

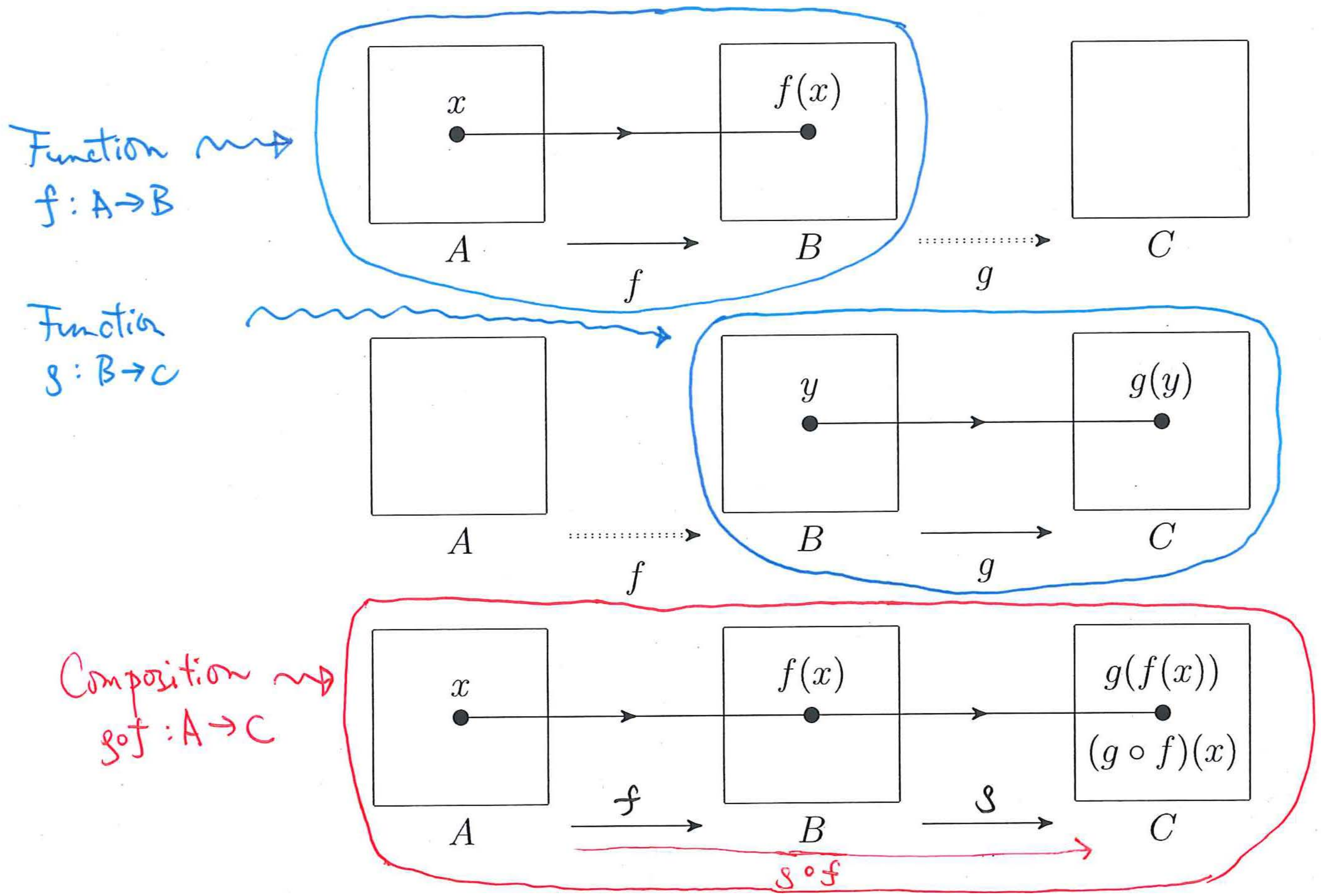
Out of two functions, the range of one being the same as the domain of the other, we may construct a third function

Definition.

Let A, B, C be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$ be functions.

Define the function $g \circ f : A \longrightarrow C$ by $(g \circ f)(x) = g(f(x))$ for any $x \in A$.

$g \circ f$ is called the **composition** *of the functions f, g.*



Example of composition.

The function '$x \longmapsto x^x$ for any $x \in (0, +\infty)$' with domain $(0, +\infty)$ and range $\mathbb{R}$ is the composition $g \circ f$ in which:

- $f : (0, +\infty) \to \mathbb{R}$ is the function given by $f(x) = x\ln(x)$ for any $x \in (0, +\infty)$, and
- $g : \mathbb{R} \to \mathbb{R}$ is the function given by $g(y) = \exp(y)$ for any $y \in \mathbb{R}$.

Lemma (1). (Associativity of composition.)
Suppose A, B, C, D be sets, and

$$f : A \longrightarrow B, \quad g : B \longrightarrow C, \quad h : C \longrightarrow D$$

are functions.
Then $(h \circ g) \circ f = h \circ (g \circ f)$ as functions.

Remark. Hence there is no ambiguity when we refer to $(h \circ g) \circ f$ (and $h \circ (g \circ f)$) as $h \circ g \circ f$.

Proof of Lemma (1).
Let A, B, C, D be sets, and $f : A \longrightarrow B$, $g : B \longrightarrow C$, $h : C \longrightarrow D$ be functions.
Note that $(h \circ g) \circ f$, $h \circ (g \circ f)$ have the same domain, namely, A.
Also note that $(h \circ g) \circ f$, $h \circ (g \circ f)$ have the same range, namely, D.
[We want to verify: For any $x \in A$, $((h \circ g) \circ f)(x) = (h \circ (g \circ f))(x)$.]

Pick any $x \in A$.
$((h \circ g) \circ f)(x) = (h \circ g)(f(x)) = h(g(f(x)))$.
$(h \circ (g \circ f))(x) = h((g \circ f)(x)) = h(g(f(x)))$.
Then $((h \circ g) \circ f)(x) = (h \circ (g \circ f))(x)$.
It follows that $(h \circ g) \circ f = h \circ (g \circ f)$ as functions. $\square$

6. **Identity function, inclusion function, restrictions and extensions.**

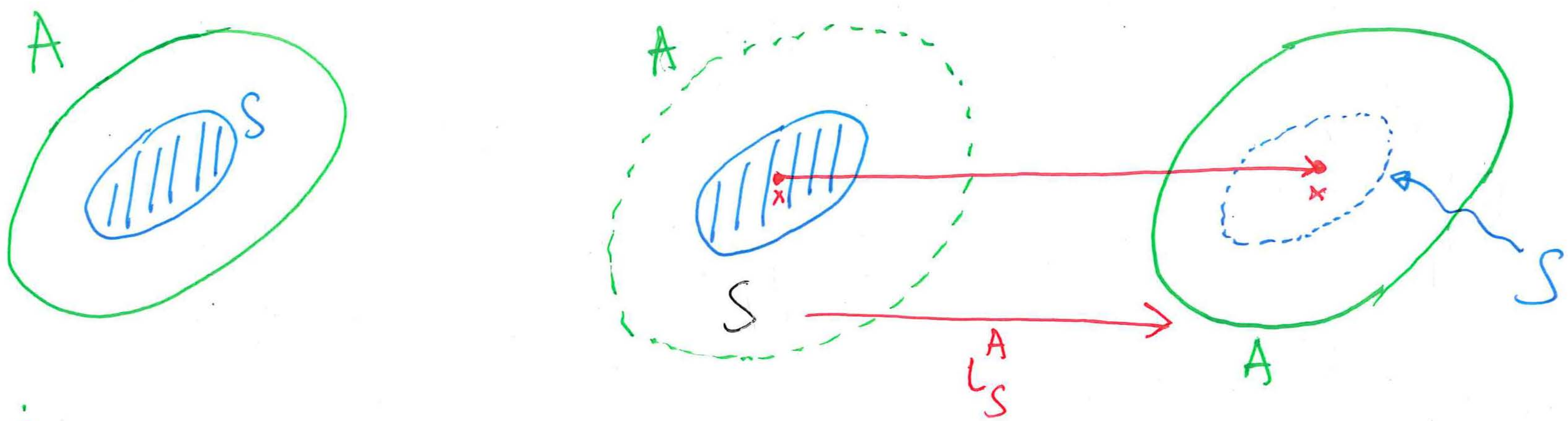
Here are the formal definitions (in terms of set language) of several miscellaneous notions used in various ocassions.

Definition.

Let A be a set.

(a) *Define the function* $\mathsf{id}_A : A \longrightarrow A$ *by* $\mathsf{id}_A(x) = x$ *for any* $x \in A$. id_A *is called* **the identity function on** A.

(b) *Let S be a subset of A. Define the function* $\iota_S^A : S \longrightarrow A$ *by* $\iota_S^A(x) = x$ *for any* $x \in S$. ι_S^A *is called the* **inclusion function of S into A.**



Reminder :

In general, ι_S^A is not the same as the identity function on A.

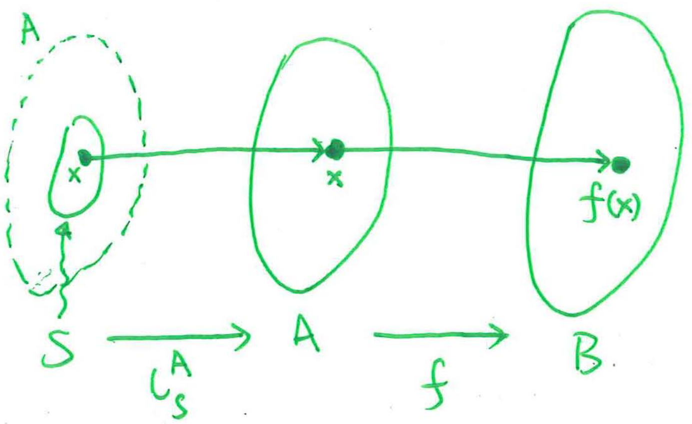
In fact, $\iota_S^A = \mathrm{id}_A$ as functions iff $S = A$.

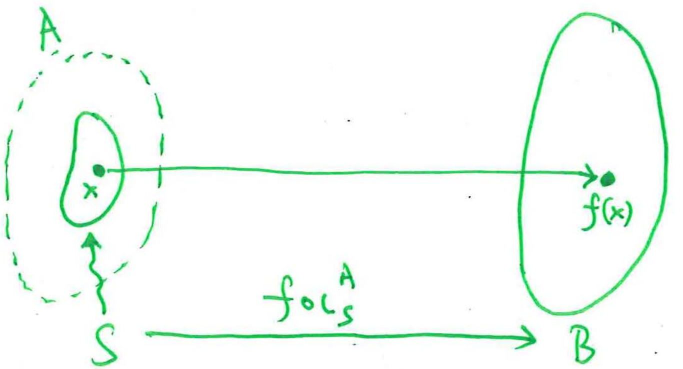
Definition.

Let A, B be sets, and $f : A \longrightarrow B$ be a function.

(a) *Let S be a subset of A. The function $f \circ \iota_S^A : S \longrightarrow B$ is called the* **restriction of f to S**. *It is denoted by $f|_S$.*

(b) *Let H be a set which contains A as a subset, K be a set which contains B as a subset. Suppose $g : H \longrightarrow K$ be a function which satisfies $g \circ \iota_A^H = \iota_B^K \circ f$. Then g is called an* **extension of f**.





Examples of restrictions from one-variable calculus.

① $\tan|_{(-\frac{\pi}{2}, \frac{\pi}{2})} : (-\frac{\pi}{2}, \frac{\pi}{2}) \to \mathbb{R}$

is the function obtained by 'restricting' the tangent function to the interval $(-\frac{\pi}{2}, \frac{\pi}{2})$.

② $\sin|_{[-\frac{\pi}{2}, \frac{\pi}{2}]} : [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{R}$

is the function obtained by 'restricting' the sine function to the interval $[-\frac{\pi}{2}, \frac{\pi}{2}]$.

Why relevant? Such 'restrictions' are useful when we want to introduce 'arctangent', 'arcsine' respectively.

7. **Graphs of functions.**

To generalize the notion of graphs of functions, we need bring in the notion of cartesian products for two arbitrary sets.

Definition.

Let A, B be sets, and $f : A \longrightarrow B$ be a function. Define $G = \{(x, f(x)) \mid x \in A\}$.

G is called the **graph of the function** *f. Note that $G \subset A \times B$.*

Lemma (2). (Equality of functions and equality of graphs.)

Let A, B be sets, and $f_1, f_2 : A \longrightarrow B$ be functions. Suppose G_1, G_2 are the respective graphs of f_1, f_2. Then f_1 is equal to f_2 as functions iff $G_1 = G_2$.

Proof of Lemma (2). Exercise in set language.

8. **‘Coordinate plane diagrams’.**

We may visualize a function, displaying its graph, by its ‘**coordinate plane diagram**’.

We illustrate the idea with the example below:

Let $A = \{m, n, p, q, r, s, t, ...\}$, $B = \{c, d, e, g, h, ...\}$, and $f : A \longrightarrow B$ *be defined by*

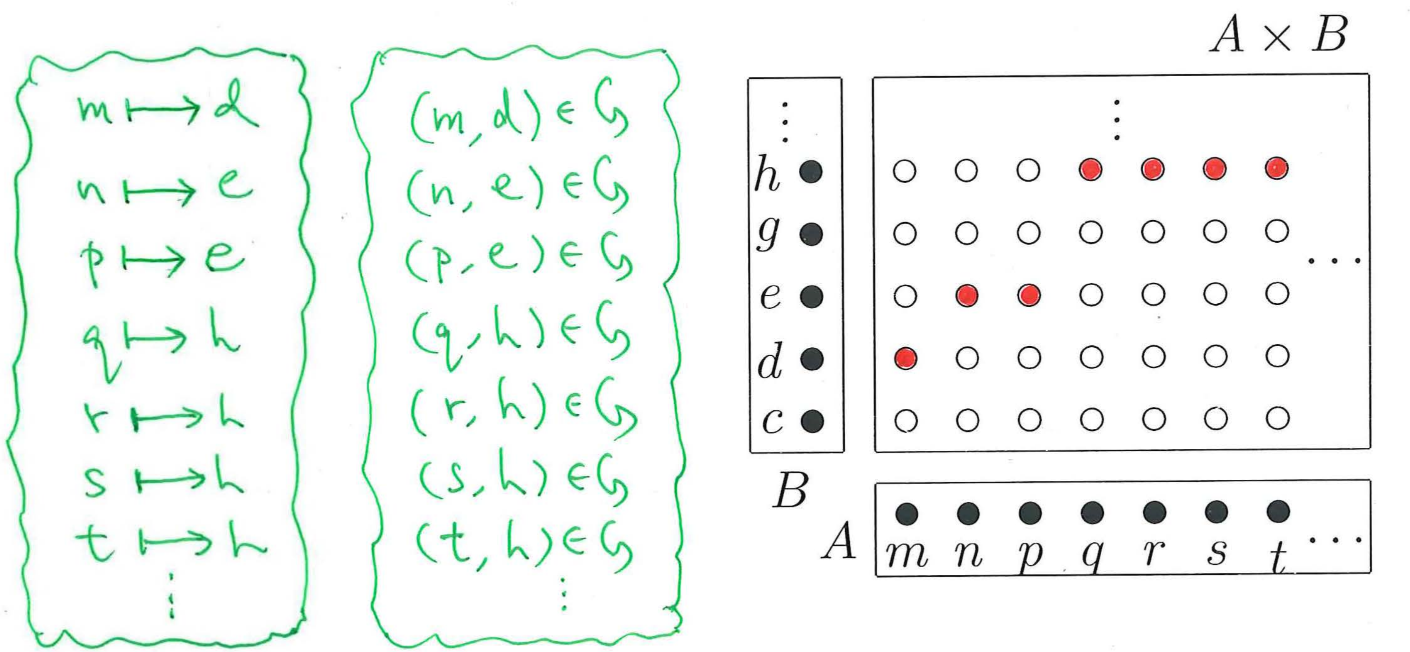
$$f(m) = d, f(n) = e, f(p) = e, f(q) = h, f(r) = h, f(s) = h, f(t) = h, \cdots .$$

By definition, the graph of f *is the set*

$$G = \{(m, d), (n, e), (p, e), (q, h), (r, h), (s, h), (t, h), \cdots\}$$

.

We draw the ‘coordinate plane diagram’ of the function f *as:*



9. **‘Blobs-and-arrows diagram’ versus ‘coordinate plane diagram’.**

Depending on how we like the ‘information’ concerned with a given function $f : A \longrightarrow B$ is presented, we may draw its ‘**coordinate plane diagram**’ or its ‘**blobs-and-arrows diagram**’.

Each has its own advantage.

The two diagrams may be converted from one to the other in a systematic way.

Illustration:

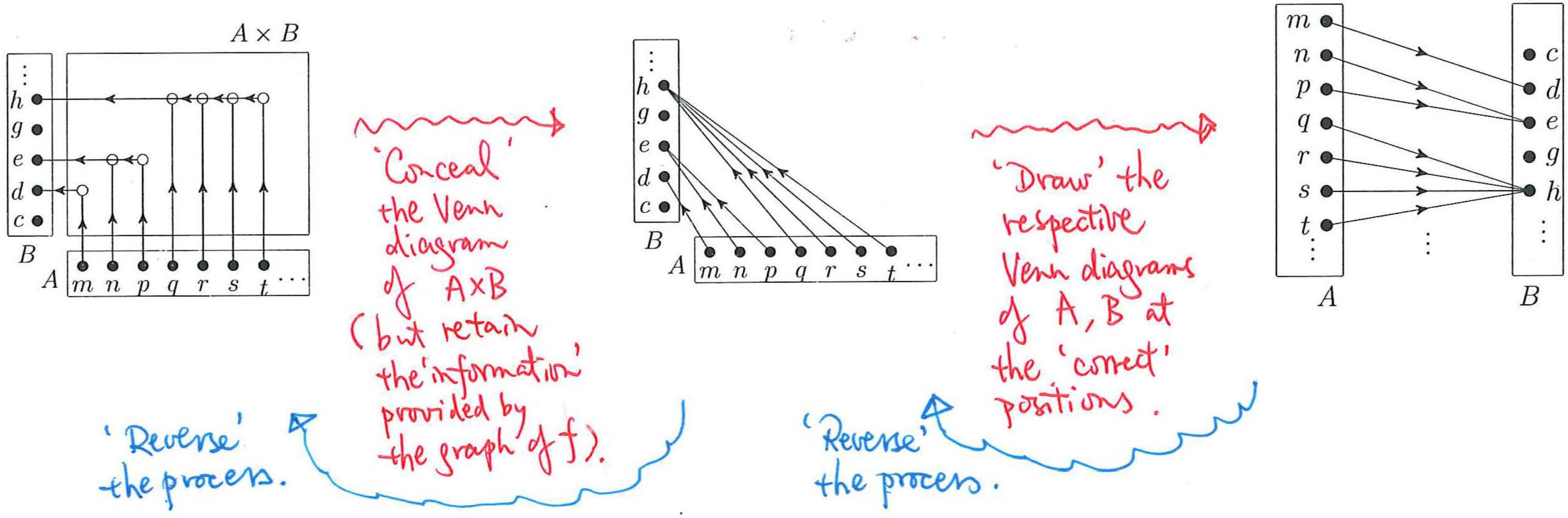
Let $A = \{m, n, p, q, r, s, t, ...\}$, $B = \{c, d, e, g, h, ...\}$, *and* $f : A \longrightarrow B$ *be defined by*

$$f(m) = d, f(n) = e, f(p) = e, f(q) = h, f(r) = h, f(s) = h, f(t) = h, \cdots .$$

(a) *‘Coordinate plane diagram’ of f:*

(b) *In-between the two kinds of diagrams:*

(c) *‘Blobs-and-arrows diagram’ of f:*



10. **Basic examples of functions in school maths and beyond.**

    We have encountered various examples of functions in school mathematics and in basic MATH courses.

    (a) **Polynomial functions with real coefficients.**
    (b) **Rational functions with real coefficients.**
    (c) **'Algebraic functions' in school maths.**
    (d) **Elementary transcendental functions.**
    (e) **'Multivariable functions' in multivariable calculus.**
    (f) **Functions of one complex variable.**
    (g) **Infinite sequences and families.**
    (h) **'Algebraic operations' for algebraic structures.**
    (i) **Linear transformations and 'transformation for various algebraic structures'.**
    (j) **Various 'operations' in calculus and beyond.**