MATH1050 Answers to Examples: Logical connectives, quantifiers, negations and dis-proofs.

1. (a) $[(P \to Q) \wedge P] \to Q$ is true irrespective of the truth values of P, Q. Hence it is a tautology.
   (b) $[(P \to Q) \wedge Q] \to P$ is a contingent statement. It is true when P, Q are both true. It is false when P is false and Q is true.

2. (a) The statements $P \to (Q \wedge R)$, $(P \to Q) \wedge (P \to R)$ are logically equivalent.
   (b) The statements $P \to (Q \to R)$, $(P \wedge Q) \to R$ are logically equivalent.
   (c) The statements $P \to (Q \vee R)$, $(P \to Q) \vee (P \to R)$ are logically equivalent.
   (d) The statements $(P \vee Q) \to R$, $(P \to R) \wedge (Q \to R)$ are logically equivalent.
   (e) The statements $(P \to Q) \to R$, $P \to (Q \to R)$ are not logically equivalent.
   (f) The statements $P \to (Q \vee R)$, $[P \wedge (\sim Q)] \to R$ are logically equivalent.

3. (a) The statement $(P \to R) \to [(P \to Q) \wedge (Q \to R)]$ is neither a tautology nor a contradiction; it is a contingent statement.
   (b) The statement $[P \to (P \to Q)] \to (P \to Q)$ is a tautology.
   (c) The statement $(P \to R) \to [(P \wedge Q) \to R)]$ is a tautology.
   (d) The statement $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$ is a tautology.
   (e) The statement $[(P \to Q) \wedge (Q \to R) \wedge (R \to P)] \to (Q \to P)$ is a tautology.
   (f) The statement $(P \to R) \to [(P \to Q) \vee (Q \to R)]$ is a tautology.
   (g) The statement $(P \to Q) \to [(Q \to R) \vee (P \wedge R)]$ is neither a tautology nor a contradiction; it is a contingent statement.
   (h) The statement $(P \to Q) \to [(P \to R) \vee (Q \to R)]$ is neither a tautology nor a contradiction; it is a contingent statement.

4. (a) There exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exists some $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$ and $|f(x) - \ell| \geq \varepsilon$.
   (b) For any $\ell \in \mathbb{R}$, there exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exists some $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$ and $|f(x) - \ell| \geq \varepsilon$.
   (c) There exists some $a \in \mathbb{R}$ such that for any $\ell \in \mathbb{R}$, there exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$ there exists some $x \in \mathbb{R}$ such that $0 < |x - a| < \delta$ and $|f(x) - \ell| \geq \varepsilon$.
   (d) There exist some $a \in \mathbb{R}$, $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exists some $x \in \mathbb{R}$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$.
   (e) There exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$ there exists some $x, a \in \mathbb{R}$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$.
   (f) There exists some $\varepsilon \in (0, +\infty)$ such that for any $\delta \in (0, +\infty)$, there exist some $x, a \in \mathbb{R}$ such that $|x - a| < \delta$ and $|f(x) - f(a)| \geq \varepsilon$.

   Remark. The given statement to be negated is:

   ‘*For any* $a \in \mathbb{R}$, *for any* $\varepsilon \in (0, +\infty)$, *there exists some* $\delta \in (0, +\infty)$ *independent of the choice of* a *such that for any* $x \in \mathbb{R}$, *if* $|x - a| < \delta$ *then* $|f(x) - f(a)| < \varepsilon$.’

   In light of the presence of the words ‘*independent of the choice of* a’ appending ‘there exists some δ’, the given statement should be understood as:

   ‘*For any* $\varepsilon \in (0, +\infty)$, *there exists some* $\delta \in (0, +\infty)$ *such that for any* $a \in \mathbb{R}$, *for any* $x \in \mathbb{R}$, *if* $|x - a| < \delta$ *then* $|f(x) - f(a)| < \varepsilon$.’

5. (a) There exist some (4×4)-matrices with real entries A, B such that $A^2 + B^2 = AB$ and (for any $p, q \in \mathbb{R}$, $pA + qB \neq BA$).
   (b) For any (4×4)-matrices with real entries A, B, there exists some $p, q \in \mathbb{R}$ such that $pA + qB = AB$ and $(pA^2 + qB^2 \neq A^2B^2$ or $pA^3 + qB^3 \neq B^2A^2)$.
   (c) There exist some (4×4)-matrices with real entries A, B such that $A + B$ is non-singular and (there exists some $p \in \mathbb{R}$ such that for any $q \in \mathbb{R}$, $pA + qB \neq AB$).

(d) For any (4×4)-matrices with real entries A, B, (there exist some $p, q \in \mathbb{R}$ such that $pA + qB \neq AB$) or (there exist some $s, t \in \mathbb{R}$ such that $sA^2 + tB^2 \neq A^2B^2$.

(e) There exist some (4×4)-matrices with real entries A, B such that (for any $p, q \in \mathbb{R}$, $pA + qB$ is singular) and (there exist some $r, s \in \mathbb{R}$ such that $rA^2 + sB^2$ is non-singular).

(f) There exist some (4×4)-matrices with real entries A, B such that (for any $p, q \in \mathbb{R}$, if $pA + qB$ is singular then $pA^2 + qB^2$ is non-singular) and (there exist some $r, s \in \mathbb{R}$ such that $rA^2 + sB^2$ is singular and $rA + sB$ is non-singular).

6. (a) False.
   (b) True.
   (c) True.
   (d) True.
   (e) True.
   (f) False.
   (g) False.
   (h) False.
   (i) False.
   (j) False.
   (k) False.
   (l) True.

7. (a) True.
   (b) True.
   (c) False.
   (d) True.
   (e) False.
   (f) True.
   (g) False.
   (h) False.
   (i) True.
   (j) True.
   (k) True.

   Hint. Start with a concrete prime number, say, 5. Ask: Is $(\sqrt{5})^{\sqrt{5}}$ rational or irrational? If it is irrational, how about $[(\sqrt{5})^{\sqrt{5}}]^{\sqrt{5}}$?

   (l) False.
   (m) False.
   (n) False.
   (o) False.
   (p) False.
   (q) False.
   (r) True.
   (s) False.
   (t) True.