MATH1050 Examples: Set operations.

1. Prove the statements below, with reference to the definitions of set equality, subset relation, intersection, union, complement, where appropriate.

   (a) *Suppose* A, B *are sets. Then* $A \cap B \subset A$.

   (b) *Suppose* A, B *are sets. Then* $A \subset A \cup B$.

   (c) *Suppose* A, B, C *are sets. Then* $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

   (d) *Suppose* A, B, C *are sets. Then* $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

   (e) *Suppose* A, B, C *are sets. Then* $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$.

   (f) *Suppose* A, B, C *are sets. Then* $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$.

2. Prove the statements below. *Where appropriate, you may apply results concerned with the properties of the set operations 'intersection', 'union' and 'complement'.*

   (a) *Suppose* A, B *are sets. Then* $A \triangle B = (A \cup B) \backslash (A \cap B)$.

   (b) *Suppose* A, B *are sets. Then* $A \triangle B = B \triangle A$.

   (c) *Suppose* A, B, C *are sets. Then* $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

   (d) *Suppose* A, B, C *be sets. Then* $A \cap (B \triangle C) = (A \cap B) \triangle (A \cap C)$.

3. Let A, B be sets. Prove that the statements below are logically equivalent, with reference to the definitions of set equality, subset relation, intersection, union, complement, where appropriate:

   (I) $A \subset B$. (II) $A \cap B = A$. (III) $A \cup B = B$.

4. (a) Prove the statements below, with reference to the definitions of set equality, subset relation, intersection, union, complement, where appropriate.

   i. *Let* A, B, T *be sets. Suppose* $A \subset T$ *and* $B \subset T$*. Then* $A \cup B \subset T$.

   ii. *Let* A, B, T *be sets. Suppose* $A \subset T$ *or* $B \subset T$*. Then* $A \cap B \subset T$.

   (b) Consider each of the statements below. For each of them, determine whether it is true or false. Justify your answer by giving a proof or constructing a counter-example where appropriate.

   i. *Let* A, B, T *be sets. Suppose* $A \cup B \subset T$*. Then* $A \subset T$ *and* $B \subset T$.

   ii. *Let* A, B, T *be sets. Suppose* $A \cap B \subset T$*. Then* $A \subset T$ *or* $B \subset T$.

   iii. *Let* A, B, T *be sets. Suppose* $A \subset T$ *or* $B \subset T$*. Then* $A \cup B \subset T$.

5. Consider each of the statements below. In each case, determine whether it is true or false. Justify your answer by giving an appropriate argument.

   (a) *Let* A, B, C *be sets. Suppose* $A \cup (B \cap C) = (A \cup B) \cap C$*. Then* $A \subset C$.

   (b) *Suppose* A, B, C *are sets. Then* $A \backslash (B \backslash C) = (A \backslash B) \backslash C$.

   (c) *Suppose* A, B *are sets. Then* $(A \cup B) \backslash A = B \backslash (A \cap B)$.

   (d) *Suppose* A, B, C *are sets. Then* $A \backslash (B \backslash C) = (A \backslash B) \cup (A \cap C)$.

   (e) *Let* A, B, C *be sets. Suppose* $A \subset B$*. Then* $C \backslash A \subset C \backslash B$.

   (f) *Let* A, B, C *be sets. Suppose* $A \subset B$ *and* $A \not\subset C$*. Then* $B \not\subset C$.

   (g) *Let* A, B, C *be non-empty sets. Suppose* $A \subset B$ *and* $B \not\subset C$*. Then* $A \not\subset C$.

   (h) *Let* A, B, C *be non-empty sets. Suppose* $A \subset B$ *and* $B \not\subset C$*. Then* $A \not\subset C$.

   (i) *Suppose* A, B, C *are sets. Then* $A \cup (B \triangle C) = (A \triangle B) \cup (A \triangle C)$.

   (j) *Suppose* A, B, C *are sets. Then* $A \cap (B \triangle C) = (A \triangle B) \cap (A \triangle C)$.

6. (a) Prove the statements below, with reference to the definitions of set equality, subset relation, intersection, union, complement, where appropriate.

   i. *Let* E *be a set, and* A, B *be subsets of* E*. Suppose* $A \subset B$*. Then* $E \backslash B \subset E \backslash A$.

ii. *Let E be a set, and A, B be subsets of E. Suppose $A \subsetneqq B$. Then $E \backslash B \subsetneqq E \backslash A$.*

(b) Consider each of the statements below. For each of them, determine whether it is true or false. Justify your answer by giving a proof or constructing a counter-example where appropriate.

i. *Let A, B, E be a set. Suppose $A \subset B$. Then $E \backslash B \subset E \backslash A$.*

ii. *Let A, B, E be a set. Suppose $A \subsetneqq B$. Then $E \backslash B \subsetneqq E \backslash A$.*

7. (a) Consider each of the statements below. For each of them, determine whether it is true or false. Justify your answer by giving a proof or constructing a counter-example where appropriate.

i. *Suppose A, B are sets. Then $B \backslash (B \backslash A) \subset A$.*

ii. *Suppose A, B are sets. Then $A \subset B \backslash (B \backslash A)$.*

(b) Prove the statements below:

i. *Suppose A, B be sets. Then $A \subset B \backslash (B \backslash A)$ iff $A \subset B$.*

ii. *Suppose A, B be sets. Then $B \backslash (B \backslash A) = A$ iff $A \subset B$.*

iii. *Suppose A, B be sets. Then $B \backslash (B \backslash A) \subsetneqq A$ iff $A \not\subset B$.*

8. (a) Consider each of the statements below. For each of them, construct an appropriate counter-example to illustrate that it is false.

i. *Let A, B be sets. Suppose $A \cap B = \emptyset$. Then $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$.*

ii. *Let A, B be sets. Suppose A, B are non-empty. Then $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B) \cup \mathfrak{P}(A \cap B)$.*

(b) Prove the statements below:

i. *Let A, B be sets. Suppose $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$. Then ($A \subset B$ or $B \subset A$).*

ii. *Let A, B be sets. Suppose ($A \subset B$ or $B \subset A$). Then $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$.*

9. Prove the statements below:

(a) *Suppose A, B are sets. Then $\mathfrak{P}(A \backslash B) \subset (\mathfrak{P}(A) \backslash \mathfrak{P}(B)) \cup \{\emptyset\}$.*

(b) *Let A, B be sets. Suppose ($A \subset B$ or $A \cap B = \emptyset$). Then $\mathfrak{P}(A) \backslash \mathfrak{P}(B) \subset \mathfrak{P}(A \backslash B)$.*

(c) *Let A, B be sets. Suppose $\mathfrak{P}(A) \backslash \mathfrak{P}(B) \subset \mathfrak{P}(A \backslash B)$. Then ($A \subset B$ or $A \cap B = \emptyset$).*

10. Dis-prove each of the statements below.

(a) *Suppose A, B are sets. Then $\mathfrak{P}(A \triangle B) \subset (\mathfrak{P}(A) \triangle \mathfrak{P}(B)) \cup \{\emptyset\}$.*

(b) *Suppose A, B are sets. Then $\mathfrak{P}(A \triangle B) \not\subset (\mathfrak{P}(A) \triangle \mathfrak{P}(B)) \cup \{\emptyset\}$.*

Remark. Note that these two statements are not negations of each other. (The statement $\sim((\forall x)(\forall y)P(x, y))$ is not the equivalent to the statement $(\forall x)(\forall y)(\sim P(x, y))$.)

11. Let M be a set, and $\{A_n\}_{n=0}^{\infty}$, $\{B_n\}_{n=0}^{\infty}$ be infinite sequences of subsets of M.

Define

$$G = \{x \in M : x \in A_n \text{ for any } n \in \mathbb{N}\}, \qquad H = \{x \in M : x \in A_n \text{ for some } n \in \mathbb{N}\},$$
$$I = \{x \in M : x \in B_n \text{ for any } n \in \mathbb{N}\}, \qquad J = \{x \in M : x \in B_n \text{ for some } n \in \mathbb{N}\}.$$

Prove the statements below:

(a) *Suppose $A_n \subset B_n$ for any $n \in \mathbb{N}$. Then $G \subset I$ and $H \subset J$.*

(b) *Suppose $K = \{x \in M : x \in A_m \cap B_n$ for some $m, n \in \mathbb{N}\}$. Then $K = H \cap J$.*

(c) *Suppose $L = \{x \in M : x \in A_m \cup B_n$ for any $m, n \in \mathbb{N}\}$. Then $L = G \cup I$.*