MATH1050 Examples: Logical connectives, quantifiers, negations and dis-proofs.

1. (a) Let P, Q be statements. Verify that the statement $[(P \to Q) \wedge P] \to Q$ is a tautology by drawing an appropriate truth table.
   (b) Let P, Q be statements. Consider the statement $[(P \to Q) \wedge Q] \to P$. Determine whether it is a tautology, or a contradiction, or a contingent statement. Justify your answer by drawing an appropriate truth table.

2. Let P, Q, R be statements. Consider each of the pairs of statements below. Determine whether the statements are logically equivalent. Justify your answer by drawing an appropriate truth table.

   (a) $P \to (Q \wedge R)$, $(P \to Q) \wedge (P \to R)$.
   (b) $P \to (Q \to R)$, $(P \wedge Q) \to R$.
   (c) $P \to (Q \vee R)$, $(P \to Q) \vee (P \to R)$.
   (d) $(P \vee Q) \to R$, $(P \to R) \wedge (Q \to R)$.
   (e) $(P \to Q) \to R$, $P \to (Q \to R)$.
   (f) $P \to (Q \vee R)$, $[P \wedge (\sim Q)] \to R$.

3. Let P, Q, R be statements. Consider each of the statements below. Determine whether it is a tautology or a contradiction or a contingent statement . Justify your answer by drawing an appropriate truth table.

   (a) $(P \to R) \to [(P \to Q) \wedge (Q \to R)]$
   (b) $[P \to (P \to Q)] \to (P \to Q)$
   (c) $(P \to R) \to [(P \wedge Q) \to R)]$
   (d) $[(P \to Q) \wedge (Q \to R)] \to (P \to R)$
   (e) $[(P \to Q) \wedge (Q \to R) \wedge (R \to P)] \to (Q \to P)$
   (f) $(P \to R) \to [(P \to Q) \vee (Q \to R)]$
   (g) $(P \to Q) \to [(Q \to R) \vee (P \wedge R)]$
   (h) $(P \to Q) \to [(P \to R) \vee (Q \to R)]$

4. Consider each of the statements below. (Do not worry about the mathematical content.) Write down its negation in such a way that the word '*not*' does not explicitly appear.

   (a) *For any* $\varepsilon \in (0, +\infty)$*, there exists some* $\delta \in (0, +\infty)$ *such that for any* $x \in \mathbb{R}$*, if* $0 < |x - a| < \delta$ *then* $|f(x) - \ell| < \varepsilon$.
   (b) *There exists some* $\ell \in \mathbb{R}$ *such that for any* $\varepsilon \in (0, +\infty)$*, there exists some* $\delta \in (0, +\infty)$ *such that for any* $x \in \mathbb{R}$*, if* $0 < |x - a| < \delta$ *then* $|f(x) - \ell| < \varepsilon$.
   (c) *For any* $a \in \mathbb{R}$*, there exists some* $\ell \in \mathbb{R}$ *such that for any* $\varepsilon \in (0, +\infty)$*, there exists some* $\delta \in (0, +\infty)$ *such that for any* $x \in \mathbb{R}$*, if* $0 < |x - a| < \delta$ *then* $|f(x) - \ell| < \varepsilon$.
   (d) *For any* $a \in \mathbb{R}$*, for any* $\varepsilon \in (0, +\infty)$*, there exists some* $\delta \in (0, +\infty)$ *such that for any* $x \in \mathbb{R}$*, if* $|x - a| < \delta$ *then* $|f(x) - f(a)| < \varepsilon$.
   (e) *For any* $\varepsilon \in (0, +\infty)$*, there exists some* $\delta \in (0, +\infty)$ *such that for any* $x, a \in \mathbb{R}$*, if* $|x - a| < \delta$ *then* $|f(x) - f(a)| < \varepsilon$.
   (f) *For any* $a \in \mathbb{R}$*, for any* $\varepsilon \in (0, +\infty)$*, there exists some* $\delta \in (0, +\infty)$ *independent of the choice of* a *such that for any* $x \in \mathbb{R}$*, if* $|x - a| < \delta$ *then* $|f(x) - f(a)| < \varepsilon$.

5. Consider each of the statements below. (Do not worry about the mathematical content.) Write down its negation in such a way that the word '*not*' does not explicitly appear.

   (a) *For any* (4×4)*-matrices with real entries* A, B*, if* $A^2 + B^2 = AB$ *then there exist some* $p, q \in \mathbb{R}$ *such that* $pA + qB = BA$.
   (b) *There exist some* (4×4)*-matrices with real entries* A, B *such that for any* $p, q \in \mathbb{R}$*, if* $pA + qB = AB$ *then* $(pA^2 + qB^2 = A^2B^2$ *and* $pA^3 + qB^3 = B^2A^2$.
   (c) *Let* A, B *be* (4×4)*-matrices with real entries. Suppose* $A + B$ *is non-singular. Then for any* $p \in \mathbb{R}$*, there exists some* $q \in \mathbb{R}$ *such that* $pA + qB = AB$.

(d) *There exist some (4×4)-matrix with real entries A, B such that (for any $p, q \in \mathbb{R}$, $pA + qB = AB$) and (for any $s, t \in \mathbb{R}$, $sA^2 + tB^2 = A^2B^2$).*

(e) *Let A, B be (4×4)-matrices with real entries. Suppose that for any $p, q \in \mathbb{R}$, $pA + qB$ is singular. Then for any $r, s \in \mathbb{R}$, $rA^2 + sB^2$ is singular.*

(f) *Let A, B be (4×4)-matrices with real entries. Suppose that for any $p, q \in \mathbb{R}$, if $pA + qB$ is singular then $pA^2 + qB^2$ is non-singular. Then for any $r, s \in \mathbb{R}$, if $rA^2 + sB^2$ is singular then $rA + sB$ is non-singular.*

6. Consider each of the statements below. For each of them, determine whether it is true or false. Justify your answer by giving an appropriate argument.

(a) *Let $a, b, c, d \in \mathbb{R}$. Suppose $a > b$ and $c > d$. Then $a - c > b - d$.*

(b) *Let n be a positive integer. Let x, y be distinct positive real numbers. $x^{2n} + y^{2n} > x^{2n-1}y + xy^{2n-1}$.*

(c) *There exist some $z, w \in \mathbb{C}$ such that $z^4 = w^4 = -1$ and $z - w \in \mathbb{R}\backslash\{0\}$.*

(d) *There exist some $x, y \in \mathbb{R}$ such that $(x + y)^2 = x^2 + y^2$.*

(e) *There exist some $x, y \in \mathbb{R}\backslash\{0\}$ such that $x^3 + y^3 = (x + y)^3$.*

(f) *There exist some $x, y \in \mathbb{R}\backslash\{0\}$ such that $x^4 + y^4 = (x + y)^4$.*

(g) *There exists some $x \in \mathbb{R}$ such that $x^8 + x^4 + 1 = 2x^2$.*

(h) *There exist some $a, b \in \mathbb{R}\backslash\{0\}$ such that $\sqrt{a^2 + b^2} = \sqrt[3]{a^3 + b^3}$.*

(i) *There exist some $x, y \in \mathbb{R}$ such that $|x^2 + iy^2| < |xy|$.*

(j) *There exists some $z \in \mathbb{C}\backslash\{0\}$ such that $\left|z + \frac{1}{\bar{z}}\right| < \left|z - \frac{1}{\bar{z}}\right|$.*

(k) *There exist some $x, y, p, s, t \in \mathbb{R}$ such that $|x - p| \leq s$ and $|y + p| \leq t$ and $|x + y| > s + t$.*

(l) *For any $\alpha \in \mathbb{C}\backslash\{0\}$, if $\alpha^2/\bar{\alpha}^2$ is a positive real number then (α is real or α is purely imaginary).*

7. Consider each of the statements below. For each of them, determine whether it is true or false. Justify your answer by giving an appropriate argument.

(a) *Let $x, n \in \mathbb{Z}$. Suppose x is divisible by n. Then for any $y \in \mathbb{Z}$, $(x + y)^3 + (x - y)^3$ is divisible by $2n$.*

(b) *Let $m, n \in \mathbb{Z}$. Suppose $m \equiv 1(\text{mod } 2)$ and $n \equiv 3(\text{mod } 4)$. Then $m^2 + n$ is divisible by 4.*

(c) *There exists some $x \in \mathbb{Z}$ such that $x \equiv 5(\text{mod } 14)$ and $x \equiv 3(\text{mod } 21)$.*

(d) *Let p, q be distinct positive prime numbers. $\dfrac{p + q}{pq}$ is not an integer.*

(e) *There exists some $n \in \mathbb{Z}$ such that $n^3 + n^2 + 2n$ is odd.*

(f) *Let x be a positive real number. Suppose x is an irrational number. Then, for any $n \in \mathbb{N}$, $t \in \mathbb{Q}$, if $n \geq 2$ then $\sqrt[n]{x + t^2}$ is an irrational number.*

(g) *Let $s, t \in \mathbb{R}$. Suppose s, t are distinct irrational numbers. Then st is an irrational number.*

(h) *Let $x, y, z \in \mathbb{N}$. Suppose $x > y > z$ and x is not divisible by y and y is not divisible by z. Then x is not divisible by z.*

(i) *There exist some $a, b \in \mathbb{R}$ such that a is irrational and b, a^b are rational.*

(j) *There exist some $a, b \in \mathbb{R}$ such that b is irrational and a, a^b are rational.*

(k) *There exist some $a, b \in \mathbb{R}$ such that a, b are irrational and a^b is rational.*

(l) *Suppose $n \in \mathbb{N}$. Then $\gcd(n, n + 2) = 2$.*

(m) *Let $x, y, z \in \mathbb{Z}$. Suppose $\gcd(x, y) > 1$ and $\gcd(y, z) > 1$. Then $\gcd(x, z) > 1$.*

(n) *Let $m, n, k \in \mathbb{Z}$. Suppose $m + n$ is divisible by k. Then m is divisible by k or n is divisible by k.*

(o) *There exists some $m, n \in \mathbb{Z}$ such that $m - n$ is divisible by 2 and $m^2 - n^2$ is not divisible by 4.*

(p) *There exists some $n \in \mathbb{N}\backslash\{0, 1, 2, 3\}$ such that n is even and $2^n - 1$ is a prime number.*

(q) *Let $x, y, z \in \mathbb{N}$. Suppose $x > yz$ and $y > z$. Further suppose x is divisible by y and x is divisible by z. Then x is divisible by yz.*

(r) *Let $a, b \in \mathbb{Z}$. Suppose a is even and b is odd. Then $a^2 + 2b^2$ is not divisible by 4.*

(s) here exist some distinct positive prime numbers p, q such that $\sqrt{p} + \sqrt{q}$ is rational.

(t) *There exists some $n \in \mathbb{N}$ such that n is a prime number and $n > 2^{100}$.*