MATH1050 Examples: Subset relation.

1. Let $A = \{x \in \mathbb{R} : x^2 - 2x - 3 \leq 0\}$, $B = \{x \in \mathbb{R} : -1 \leq x \leq 3\}$.
   Prove that $A = B$.

2. Let $A = \{x \in \mathbb{Z} : x = k^4 \text{ for some } k \in \mathbb{Z}\}$, $B = \{x \in \mathbb{Z} : x = k^2 \text{ for some } k \in \mathbb{Z}\}$.
   (a) Prove that $A \subset B$.
   (b) Prove that $B \not\subset A$.

3. Let $A = \{x \mid x = r^6 \text{ for some } r \in \mathbb{Q}\}$, $B = \{x \mid x = r^2 \text{ for some } r \in \mathbb{Q}\}$.
   (a) Prove that $A \subset B$.
   (b) Prove that $B \not\subset A$.

4. Let $A = \{x \mid x = 54m^6 \text{ for some } m \in \mathbb{Z}\}$, $B = \{x \mid x = 2m^3 \text{ for some } m \in \mathbb{Z}\}$.
   (a) Prove that $A \subset B$.
   (b) Prove that $B \not\subset A$.

5. *In this question you may take for granted the validity of the statements below:*
   ($\sharp$) *Suppose p, q be distinct positive prime numbers. Then $\sqrt{pq}$ is irrational.*
   ($\flat$) *$\sqrt{6}$ is an irrational number.*

   Let $A = \{x \mid x = s + t\sqrt{2} \text{ for some } s, t \in \mathbb{Z}\}$, $B = \{x \mid x = u + v\sqrt{3} \text{ for some } u, v \in \mathbb{Z}\}$.
   (a) Prove that $A \not\subset B$.
   (b) Prove that $A \cap B = \mathbb{Z}$.

6. We introduce the definition for the notion of *congruence for integers* below:
   - *Let $n \in \mathbb{N}\backslash\{0, 1\}$, and $a, b \in \mathbb{Z}$. We say a is* **congruent to** *b* **modulo** *n if $a - b$ is divisible by n. We write $a \equiv b(\text{mod } n)$.*

   Let $A = \{n \in \mathbb{Z} : n \equiv 1(\text{mod } 3)\}$, $B = \{n \in \mathbb{Z} : n \equiv 4(\text{mod } 9)\}$.
   (a) Prove that $B \subset A$.
   (b) Prove that $A \not\subset B$.

7. Let $C = \{\zeta \in \mathbb{C} : |\zeta - 1| \leq 1\}, D = \{\zeta \in \mathbb{C} : |\zeta| \leq 2\}$.
   (a) Prove that $C \subset D$.
   Remark. Make good use of the Triangle Inequality for complex numbers.
   (b) Prove that $D \not\subset C$.

8. Let $C = \{\zeta \in \mathbb{C} : \text{Re}(\zeta) \geq 0\}, D = \{\zeta \in \mathbb{C} : \text{Im}(\zeta) \geq 0\}, E = \{\zeta \in \mathbb{C} : |\zeta - 1 - i| \leq 1\}$.
   (a) Prove that $E \subset C \cap D$.
   (b) Prove that $C \not\subset D$.
   (c) Prove that $D \not\subset C$.

9. Let $A = \{z \mid z = 8^m(\cos(6m) + i\sin(6m)) \text{ for some } m \in \mathbb{Z}\}$, $B = \{z \mid z = 2^m(\cos(2m) + i\sin(2m)) \text{ for some } m \in \mathbb{Z}\}$.
   (a) Prove that $A \subset B$.
   (b) Prove that $B \not\subset A$.

10. Let $A = \{z \in \mathbb{C} : z^2 = ri \text{ for some } r \in \mathbb{R}\}$, $B = \{z \in \mathbb{C} : z^6 = ri \text{ for some } r \in \mathbb{R}\}$.

(a) Prove that $A \subset B$.

(b) Prove that $B \not\subset A$.

Remark. The *polar form for complex numbers* and *De Moivre's Theorem* are useful.

11. Let $A = \{x \in \mathbb{R} : x^2 - x \geq 0\}$, $B = \{x \in \mathbb{R} : x \leq 0\}$, $C = \{x \in \mathbb{R} : x \geq 1\}$.

    Prove that $A = B \cup C$.

12. Let $A = \{x \in \mathbb{Q} : x = r^3 \text{ for some } r \in \mathbb{Q}\}$, $B = \{x \in \mathbb{Q} : x = r^9 \text{ for some } r \in \mathbb{Q}\}$, $C = \{x \in \mathbb{Z} : x = r^3 \text{ for some } r \in \mathbb{Q}\}$, $D = \{x \in \mathbb{Q} : x = r^3 \text{ for some } r \in \mathbb{Z}\}$.

    (a) Is A a subset of $\mathbb{Q}$? Is $\mathbb{Q}$ a subset of A? Justify your answer.

    (b) Is A a subset of B? Is B a subset of A? Justify your answer.

    (c) Is A a subset of C? Is C a subset of A? Justify your answer.

    (d) Is A a subset of D? Is D a subset of A? Justify your answer.

13. Let $A = \{\zeta \mid \zeta = m^3 + n^4 i \text{ for some } m, n \in \mathbb{N}\}$, $B = \{\zeta \mid \zeta = m + n^8 i \text{ for some } m, n \in \mathbb{N}\}$.

    (a) Is it true that $A \subset B$? Justify your answer.

    (b) Is it true that $B \subset A$? Justify your answer.

14. For each integer n greater than 1, define $Z_n = \{\zeta \in \mathbb{C} : \zeta \text{ is a } n\text{-th root of unity}\}$.

    Let m, n be integers greater than 1.

    Prove the statements below:

    (a) *Suppose m is divisible by n. Then Z_n is a subset of Z_m.*

    (b) *Suppose Z_n is a subset of Z_m. Then m is divisible by n.*

    Remark. You may take for granted the validity of **Division Algorithm**:

    Let $u, v \in \mathbb{N}$. Suppose $v \neq 0$. Then there exist some $q, r \in \mathbb{N}$ such that $u = qv + r$ and $0 \leq r < v$.

Answer.

12. (a) A is a subset of $\mathbb{Q}$. $\mathbb{Q}$ is not a subset of A.
    (b) A is not a subset of B. B is a subset of A.
    (c) A is not a subset of C. C is a subset of A.
    (d) A is not a subset of D. D is a subset of A.
13. (a) A is not a subset of B.
    (b) B is not a subset of A.