0. We introduced the notions of ordered pairs and cartesian products in the Handout *Ordered pairs, ordered triples and cartesian products*, and through them we introduce the notion of functions.

   We now make use of functions to see the notions of ordered pairs and cartesian products in another way. It will suggest how we may generalize the notion of ordered pairs (of two objects) to that of ordered tuples for an arbitrary collection, and how we may generalize the notion of cartesian products (of two sets) to that of cartesian products for an arbitrary collection of sets.

   Throughout this Handout, $\mathsf{Map}(A, B)$ stands for the set of all functions from A to B. (In some set theory textbook, this set is denoted by B^A.)

1. **Ordered pairs seen as functions, and Cartesian products seen as sets of functions.**

   We start with an easy result, the proof of which is left as an exercise.

   Lemma (1).

   Let S_0, S_1 be sets. Let $g, h \in \mathsf{Map}(\{0,1\}, S_0 \cup S_1)$. Suppose $g(0), h(0) \in S_1$ and $g(1), h(1) \in S_1$. Then $g = h$ as functions iff $(g(0) = h(0)$ and $g(1) = h(1))$.

   Remark. The assumption 'Suppose $g(0), h(0) \in S_0$ and $g(1), h(1) \in S_1$' is redundant, in the sense that they play no role in the argument for Lemma (1). It is deliberately added here so as to emphasize the point of the observation that we make now. Lemma (1) suggests that:

   (a) we may identify the cartesian product $S_0 \times S_1$ as the set $\{f \in \mathsf{Map}(\{0,1\}, S_0 \cup S_1) : f(0) \in S_0$ and $f(1) \in S_1\}$, which we denote by $\prod_{j=0}^{1} S_j$;

   (b) whenever $x_0 \in S_0$ and $x_1 \in S_1$, we may identify the ordered pair (x_0, x_1) as the element of $\prod_{j=0}^{1} S_j$ which assigns 0 to x_0 and 1 to x_1 as a function from $\{0,1\}$ to $S_0 \cup S_1$.

   Lemma (1) can be generalized to a result about ordered n-tuples and Cartesian products of n sets:

   Lemma (2).

   Let n be a positive integer, and $S_0, S_1, \cdots, S_{n-1}$ be sets. Let $g, h \in \mathsf{Map}([\![0, n-1]\!], S_0 \cup S_1 \cup \cdots \cup S_{n-1})$. Suppose $g(j), h(j) \in S_j$ for each $j \in [\![0, n-1]\!]$. Then $g = h$ as functions iff $(g(j) = h(j)$ for each $j \in [\![0, n-1]\!])$.

   Remark. Lemma (2) suggests that:

   (a) we may identify the cartesian product $S_0 \times S_1 \times \cdots \times S_{n-1}$ as the set $\{f \in \mathsf{Map}([\![0, n-1]\!], S_0 \cup S_1 \cup \cdots \cup S_{n-1}) : f(j) \in S_j$ for any $j \in [\![0, n-1]\!]\}$, which we denote by $\prod_{j=0}^{n-1} S_j$;

   (b) whenever $x_j \in S_j$ for each $j \in [\![0, n-1]\!]$, we may identify the ordered n-tuple $(x_0, x_1, \cdots, x_{n-1})$ as the element of $\prod_{j=0}^{n-1} S_j$ which assigns j to x_j for each $j \in [\![0, n-1]\!]$ as a function from $[\![0, n-1]\!]$ to $S_0 \cup S_1 \cup \cdots \cup S_{n-1}$.

2. **Generalized Cartesian product.**

   Lemma (1) and Lemma (2) suggest how we may define Cartesian product for an infinite sequence of sets.

   Definition. (Generalized Cartesian product for infinite sequence of subsets of any given set.)

   Let M be a set, and $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of the set M. (For any $n \in \mathbb{N}$, S_n is a subset of M.)

   The **(generalized) Cartesian product of the infinite sequence of subsets $\{S_n\}_{n=0}^{\infty}$ of the set** M *is defined to be the set $\{f \in \mathsf{Map}(\mathbb{N}, M) \mid f(n) \in S_n$ for any $n \in \mathbb{N}\}$. It is denoted by $\prod_{n=0}^{\infty} S_n$.*

Remark. This is how the definitions may be understood heuristically. Suppose $f \in \mathsf{Map}(\mathbb{N}, M)$. By definition, f is an infinite sequence in M. For each $j \in \mathbb{N}$, we write $x_j = f(j)$. Then

$$f \in \prod_{n=0}^{\infty} S_n \quad \text{iff} \quad (f(n) \in S_n \text{ for any } n \in \mathbb{N})$$

$$\text{`iff'} \quad \text{`}x_0 \in S_0 \text{ and } x_1 \in S_1 \text{ and } x_2 \in S_2 \text{ and ...'}$$

Thus $\prod_{n=0}^{\infty} S_n$ consists, as its elements, of those objects which might be visualized as infinite sequences in M and for each of which the j-th term is an element of S_j for each $j \in \mathbb{N}$.

The notion of Cartesian products can be further generalized from that for an infinite sequence of subsets of an arbitrary set to that for a set of subsets of an arbitrary set, through the notion of families. (Refer to the Handout *Notion of functions and its pictorial visualizations* for the definition for the notion of families.)

Definition. (Generalized Cartesian product for a set of subsets of an arbitrary set.)

Let I, M be sets, and $\{S_p\}_{p\in I}$ be a family of subsets of M, indexed by I. (By definition, such a family is a function from I to $\mathfrak{P}(M)$ defined by $p \longmapsto S_p$ for each $p \in I$.)

The **(generalized) Cartesian product of the family of subsets** $\{S_p\}_{p\in I}$ **of the set** M *is defined to be the set $\{f \in \mathsf{Map}(I, M) : f(p) \in S_p$ for any $p \in I\}$. It is denoted by $\prod_{p\in I} S_p$.*

Remark. Each element of $\prod_{p\in I} S_p$ is a function, say, f, with domain I and range M, assigning each $p \in I$ to some element $f(p)$ of the set S_p.

3. **Axiom of Choice.**

We now introduce a statement, known as the Axiom of Choice, which looks intuitively obvious to be true.

Axiom of Choice. (AC1)

Let I, M be non-empty sets, and $\Phi : I \longrightarrow \mathfrak{P}(M)$ be a function. Suppose $\Phi(\alpha) \neq \emptyset$ for any $\alpha \in I$. Then there exists a function $\varphi : I \longrightarrow M$ such that $\varphi(\alpha) \in \Phi(\alpha)$ for any $\alpha \in I$.

Remark. As its name suggests, (AC1) is a statement whose validity we can only choose between *believing* and *not believing*. Many mathematicians choose to *believe* it to be true.

Re-formulation of the Axiom of Choice. (AC2)

For any non-empty set A, there exists some function $\psi : \mathfrak{P}(A)\backslash\{\emptyset\} \longrightarrow A$ such that $\psi(S) \in S$ for any $S \in \mathfrak{P}(A)\backslash\{\emptyset\}$.

Remark. Such a function ψ is called a **choice function** in the set A.

Theorem (3).

The statements (AC1), (AC2) *are logically equivalent.*

Proof.

- [(AC1)$\Longrightarrow$(AC2)?]

  Suppose (AC1) holds.

  Let A be a non-empty set. The inclusion function $\iota : \mathfrak{P}(A)\backslash\{\emptyset\} \longrightarrow \mathfrak{P}(A)$ is a function which satisfies $\iota(T) \neq \emptyset$ for any $T \in \mathfrak{P}(A)\backslash\{\emptyset\}$.

  By (AC1), there exists some function $\varphi : \mathfrak{P}(A)\backslash\{\emptyset\} \longrightarrow A$ such that $\varphi(T) \in \iota(T)$ for any $T \in \mathfrak{P}(A)\backslash\{\emptyset\}$.

  Hence, for this function $\varphi : \mathfrak{P}(A)\backslash\{\emptyset\} \longrightarrow A$, for any $T \in \mathfrak{P}(A)\backslash\{\emptyset\}$, we have $\varphi(T) \in \iota(T) = T$.

  Hence (AC2) follows.

- [(AC2)$\Longrightarrow$(AC1)?]

  Suppose (AC2) holds.

  Let I, M be non-empty sets, and $\Phi : I \longrightarrow \mathfrak{P}(M)$ be a function. Suppose $\Phi(\alpha) \neq \emptyset$ for any $\alpha \in I$.

  By (AC2), there exists some function $\psi : \mathfrak{P}(M)\backslash\{\emptyset\} \longrightarrow M$ such that $\psi(S) \in S$ for any $S \in \mathfrak{P}(M)\backslash\{\emptyset\}$.

  Define $\varphi : I \longrightarrow M$ by $\varphi(\alpha) = \psi(\Phi(\alpha))$ for any $\alpha \in I$. (For any $\alpha \in I$, $\psi(\Phi(\alpha))$ is well-defined because $\Phi(\alpha) \neq \emptyset$.)

  By definition, for any $\alpha \in I$, $\varphi(\alpha) = \psi(\Phi(\alpha)) \in \Phi(\alpha)$.

  Hence (AC1) follows.

4. **Re-formulation of the Axiom of Choice in terms of generalized cartesian products.**

   Examine the statement (AC1) again:

   Let I, M be non-empty sets, and $\Phi : I \longrightarrow \mathfrak{P}(M)$ be a function. Suppose $\Phi(\alpha) \neq \emptyset$ for any $\alpha \in I$. Then there exists a function $\varphi : I \longrightarrow M$ such that $\varphi(\alpha) \in \Phi(\alpha)$ for any $\alpha \in I$.

   What is the statement (AC1) telling us? Assuming the function Φ to assign each element of I to some non-empty subset of M, there will be a function φ through which we 'choose', for each element α of I, some element of the non-empty subset $\Phi(\alpha)$ of M, namely $\varphi(\alpha)$. By definition, such a function $\varphi : I \longrightarrow M$ is an element of the set $\prod_{\alpha\in I} \Phi(\alpha)$. With the existence of such a φ as an element of $\prod_{\alpha\in I} \Phi(\alpha)$, we see that $\prod_{\alpha\in I} \Phi(\alpha)$ is a non-empty set.

   This is something expected. We will never suspect that the cartesian product $S \times T$ of two non-empty sets S, T may be empty; we can in fact pinpoint a concrete element, say, (u, v), of $S \times T$, by first picking out an element u of S, an element v of T, and then forming the ordered pair (u, v). However, when it comes just an 'arbitrary non-empty family of non-empty sets', we have no way to explicitly pinpoint an element of the generalized cartesian product formed by this family. The existence of an element in the generalized cartesian product is the essence of the Axiom of Choice.

   Re-formulation of the Axiom of Choice, in terms of generalized cartesian products. (AC3)

   The Cartesian product of any non-empty family of non-empty sets is non-empty.

   Theorem (4).

   The statements (AC1), (AC2), (AC3) are logically equivalent.

   Proof. Exercise.

5. **Axiom of Choice, Well-ordering Principle and Zorn's Lemma.**

   Recall the Well-ordering Principle and Zorn's Lemma:

   Well-ordering Principle.

   Suppose A is a set. Then there exists some partial ordering R in A such that A is well-ordered by R.

   Zorn's Lemma.

   Let A be a set, partially ordered by T. Suppose every chain in A with respect to T is bounded above with respect to T. Then A has a maximal element with respect to T.

   These two statements are something whose validity we may choose between *believing* and *not believing*.

   It turns out that these two statements and the Axiom of Choice are the same thing.

   Theorem (5).

   The Axiom of Choice is logically equivalent to each of the Well-ordering Principle and Zorn's Lemma.

   Proof. Omitted here; refer to any standard textbook in axiomatic set theory.