

1. **Families.**

The notion of *infinite sequences of real numbers* can be generalized to the notion of families for collections of objects. This in turn provides a tool for us to generalize the notions of *intersections* and *unions*, (as well as *cartesian products*).

Definition.

Suppose I, B are sets, and $\varphi : I \rightarrow B$ is a function.

Then we say φ is a **family in B , indexed by I** .

The set I is referred to as the **index set** for this family, and the set $\{x \in B : x = \varphi(t) \text{ for some } t \in I\}$ is called the **set of all terms** for this family.

Remarks.

- (a) This definition generalizes the notion of infinite sequences of real numbers. When $I = \mathbb{N}$, a family in B is just an infinite sequence in B ; when furthermore $B = \mathbb{R}$, it is just an infinite sequence of real numbers.
- (b) We imitate the notations for infinite sequences when we regard the function φ from I to B as a family in B with index set I . We suppress the symbols φ, B , and present this family as, say, $\{x_\alpha\}_{\alpha \in I}$, in which the symbol x_α stands for $\varphi(\alpha)$ for each $\alpha \in I$.
- (c) What is the point of this definition?
It gives us the flexibility to regard the same family in B , say, $\{x_\alpha\}_{\alpha \in I}$, as a family in C which contains B as a subset, whenever it is convenient for us to do so.

Examples.

- (a) $\{(-\infty, u)\}_{u \in \mathbb{R}}$ stands for the function with domain \mathbb{R} and range, say, the set of all intervals, assigning each real number u to the open interval $(-\infty, u)$.
- (b) $\{(-|a|, |a| + 1)\}_{a \in \mathbb{Q}}$ stands for the function with domain \mathbb{Q} and range, say, the set of all intervals, assigning each rational number a to the open interval $(-|a|, |a| + 1)$.
- (c) Let p, q be real numbers.
For any real numbers a, b , not both zero, define $\ell_{(a,b)}$ by $\ell_{(a,b)} = \{(x, y) \mid x, y \in \mathbb{R} \text{ and } a(x - p) + b(y - q) = 0\}$.
 $\ell_{(a,b)}$ is the line in the infinite plane whose equation is given by $ax + by = ap + bq$.
 $\{\ell_{(a,b)}\}_{(a,b) \in \mathbb{R}^2 \setminus \{(0,0)\}}$ stands for the function with domain $\mathbb{R}^2 \setminus \{(0,0)\}$ and range, say, the set of all lines in the infinite plane, assigning each $(a, b) \in \mathbb{R}^2 \setminus \{(0,0)\}$ to the line $\ell_{(a,b)}$.
(The set of all terms of this family is the pencil of all lines passing through (p, q) .)
- (d) Let a, b, c be real numbers, not all zero.
For any real number d , define π_d by $\pi_d = \{(x, y, z) \mid x, y, z \in \mathbb{R} \text{ and } ax + by + cz = d\}$.
 π_d is the plane in the infinite space whose equation is given by $ax + by + cz = d$.
 $\{\pi_d\}_{d \in \mathbb{R}}$ stands for the function with domain \mathbb{R} and range, say, the set of all planes in the infinite space, assigning each $d \in \mathbb{R}$ to the plane π_d .
(The set of all terms of this family is the pencil of planes all parallel to the plane $ax + by + cz = 0$.)
- (e) For each complex number p , for each positive real number r , define $C(\zeta, r) = \{z \in \mathbb{C} : |z - p| = r\}$. ($C(\zeta, r)$ is the circle in the Argand plane with centre p and radius r .)
Let $p \in \mathbb{C}$.
 $\{C(p, r)\}_{r \in (0, +\infty)}$ stands for the function with domain $(0, +\infty)$ and range, say, the set of all circles in the Argand plane, assigning each positive real number to the circle $C(p, r)$.
(The set of all terms of this family is the pencil of circles concentric at p .)
- (f) For each complex number p , for each positive real number r , define $D(\zeta, r) = \{z \in \mathbb{C} : |z - p| < r\}$. ($D(\zeta, r)$ is the ‘open disc’ in the Argand plane with centre p and radius r .)
 $\{D(r, r)\}_{r \in (0, +\infty)}$ stands for the function with domain $(0, +\infty)$ and range, say, the set of all open discs in the Argand plane, assigning each positive real number to the open disc $D(r, r)$.
- (g) For each real number α , define the polynomial function $f_\alpha : \mathbb{R} \rightarrow \mathbb{R}$ by $f_\alpha(x) = x^2 + \alpha x + 1$ for any $x \in \mathbb{R}$.
 $\{f_\alpha\}_{\alpha \in \mathbb{R}}$ stands for the function with domain \mathbb{R} and range, say, the set of all polynomial functions, assigning each real number α to f_α .

2. Set operations for families of sets.

The notion of intersection and union for infinite sequences of sets, introduced in the Handout *Universal quantifier and existential quantifier*, can be immediately generalized to families of sets.

Definition.

Let M, I be sets, and $\{S_\alpha\}_{\alpha \in I}$ be a family of subsets of the set M , indexed by I . (For any $\alpha \in I$, S_α is a subset of M .)

- (1) The **intersection of the family of subsets** $\{S_\alpha\}_{\alpha \in I}$ **of the set** M is defined to be the set $\{x \in M : x \in S_\alpha \text{ for any } \alpha \in I\}$. It is denoted by $\bigcap_{\alpha \in I} S_\alpha$.
- (2) The **union of the family of subsets** $\{S_\alpha\}_{\alpha \in I}$ **of the set** M is defined to be the set $\{x \in M : x \in S_\alpha \text{ for some } \alpha \in I\}$. It is denoted by $\bigcup_{\alpha \in I} S_\alpha$.

Remark. Suppose C is a subset of $\mathfrak{P}(M)$, and we ‘index’ C by its elements to obtain the family $\{S\}_{S \in C}$. Then the intersection of the set C of subsets of the set M is the intersection of the family $\{S\}_{S \in C}$, and the union of the set C of subsets of the set M is the union of the family $\{S\}_{S \in C}$.

3. Theorem (\star) in the Handout *Universal quantifier and existential quantifier* can be generalized immediately to Theorem (\star') below. The proofs of the respective statements are similar.

Theorem (\star') .

Let M, I be sets and $\{A_\alpha\}_{\alpha \in I}$ be a family of subsets of M , indexed by I .

- (0) Suppose $I = \emptyset$. Then $\bigcap_{\alpha \in I} A_\alpha = M$ and $\bigcup_{\alpha \in I} A_\alpha = \emptyset$.
- (1) Let S be a subset of M . Suppose $S \subset A_\alpha$ for any $\alpha \in I$. Then $S \subset \bigcap_{\alpha \in I} A_\alpha$.
- (2) Let S be a subset of M . Suppose $S \subset A_\alpha$ for some $\alpha \in I$. Then $S \subset \bigcup_{\alpha \in I} A_\alpha$.
- (3) Let T be a subset of M . Suppose $A_\alpha \subset T$ for any $\alpha \in I$. Then $\bigcup_{\alpha \in I} A_\alpha \subset T$.
- (4) Let T be a subset of M . Suppose $A_\alpha \subset T$ for some $\alpha \in I$. Then $\bigcap_{\alpha \in I} A_\alpha \subset T$.
- (5) Let C be a subset of M . ($\{A_\alpha \cup C\}_{\alpha \in I}, \{A_\alpha \cap C\}_{\alpha \in I}, \{A_\alpha \setminus C\}_{\alpha \in I}, \{C \setminus A_\alpha\}_{\alpha \in I}$ are families of subsets of M .) The equalities below hold:

$$(5a) \quad \left(\bigcap_{\alpha \in I} A_\alpha \right) \cap C = \bigcap_{\alpha \in I} (A_\alpha \cap C).$$

$$(5e) \quad \left(\bigcap_{\alpha \in I} A_\alpha \right) \setminus C = \bigcap_{\alpha \in I} (A_\alpha \setminus C).$$

$$(5b) \quad \left(\bigcup_{\alpha \in I} A_\alpha \right) \cup C = \bigcup_{\alpha \in I} (A_\alpha \cup C).$$

$$(5f) \quad \left(\bigcup_{\alpha \in I} A_\alpha \right) \setminus C = \bigcup_{\alpha \in I} (A_\alpha \setminus C).$$

$$(5c) \quad \left(\bigcap_{\alpha \in I} A_\alpha \right) \cup C = \bigcap_{\alpha \in I} (A_\alpha \cup C).$$

$$(5g) \quad C \setminus \left(\bigcap_{\alpha \in I} A_\alpha \right) = \bigcup_{\alpha \in I} (C \setminus A_\alpha).$$

$$(5d) \quad \left(\bigcup_{\alpha \in I} A_\alpha \right) \cap C = \bigcup_{\alpha \in I} (A_\alpha \cap C).$$

$$(5h) \quad C \setminus \left(\bigcup_{\alpha \in I} A_\alpha \right) = \bigcap_{\alpha \in I} (C \setminus A_\alpha).$$