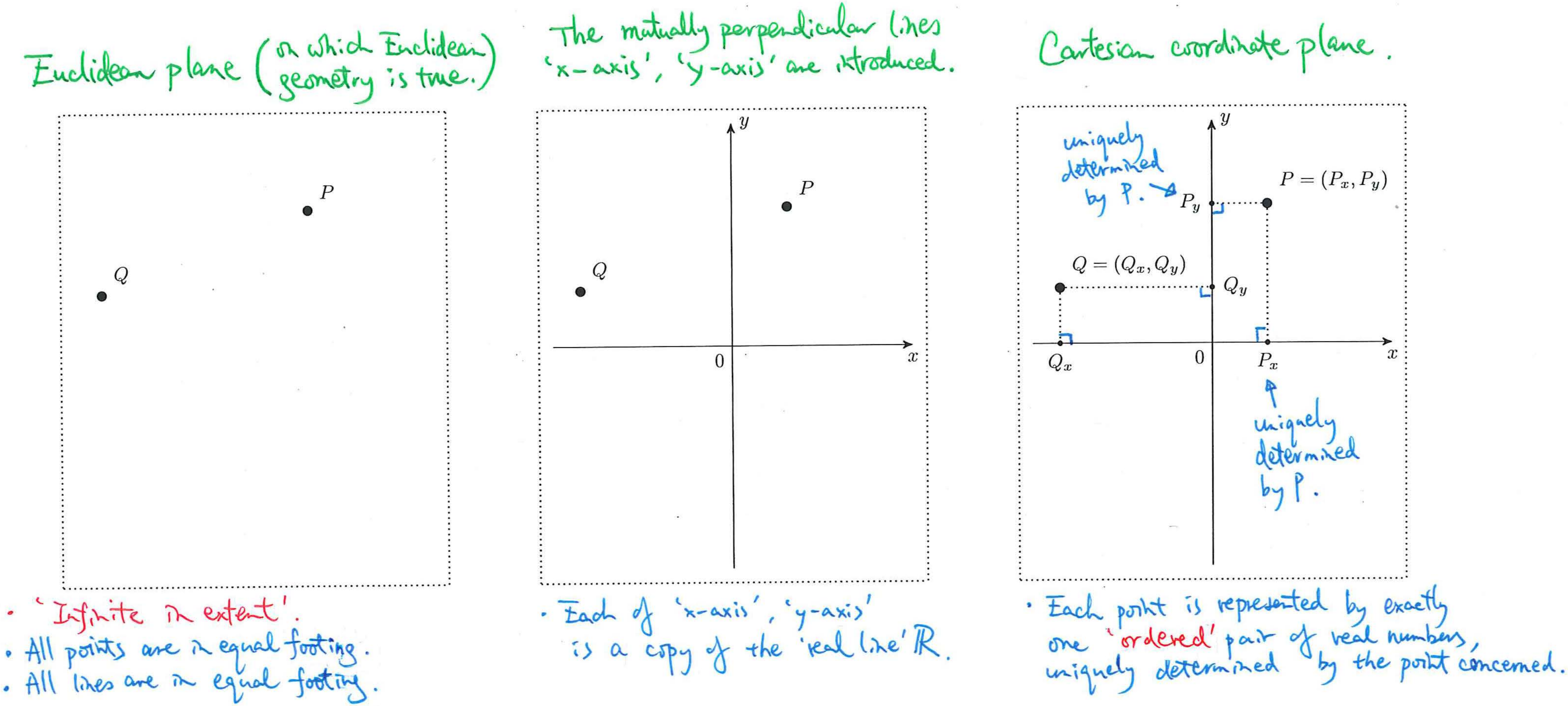
1. **Coordinate pairs and Cartesian plane in school mathematics**.

In school mathematics, we take the notions of coordinate pairs and the Cartesian (coordinate) plane for granted:



This is then generalized to coordinate triples and the Cartesian (coordinate) space, and beyond.

Here we generalize the idea above in the context of set language.

2. **Ordered-ness in set language, and Cartesian product of two sets.**

Question. *What is the essence in the notion of coordinate pairs in the plane?*

- *For any* $s, t, u, v \in \mathbb{R}$, *($(s, t) = (u, v)$ iff ($s = u$ and $t = v$)).*

This guarantees that you will not confuse the point, say, (1, −1), with the point (−1, 1).

Imagine it makes sense to talk about the object called **ordered pair**

$$(s, t)$$

of

$$s, \qquad t$$

with s, t as first and second coordinate of (s, t).

But we require '$(\cdot, \cdot)$' to obey Convention ($\sharp$) below:

($\sharp$) *For any objects* x, y, u, v, *($(x, y) = (u, v)$ iff ($x = u$ and $y = v$)).*

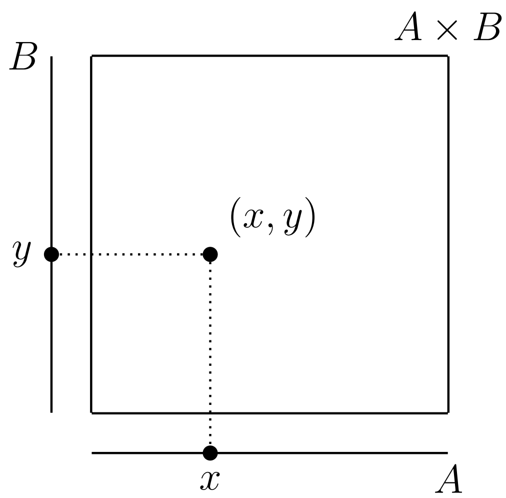
3. **Cartesian product of two sets.**

With this sense of ordered-ness in mind, it makes sense to define the notion of Cartesian product of two sets:

Definition.

Let A, B *be sets. The* **cartesian product** $A \times B$ *of the sets* A, B *is defined to be the set*

$$\{t \mid \text{ There exist some } x \in A, y \in B \text{ such that } t = (x, y)\}.$$



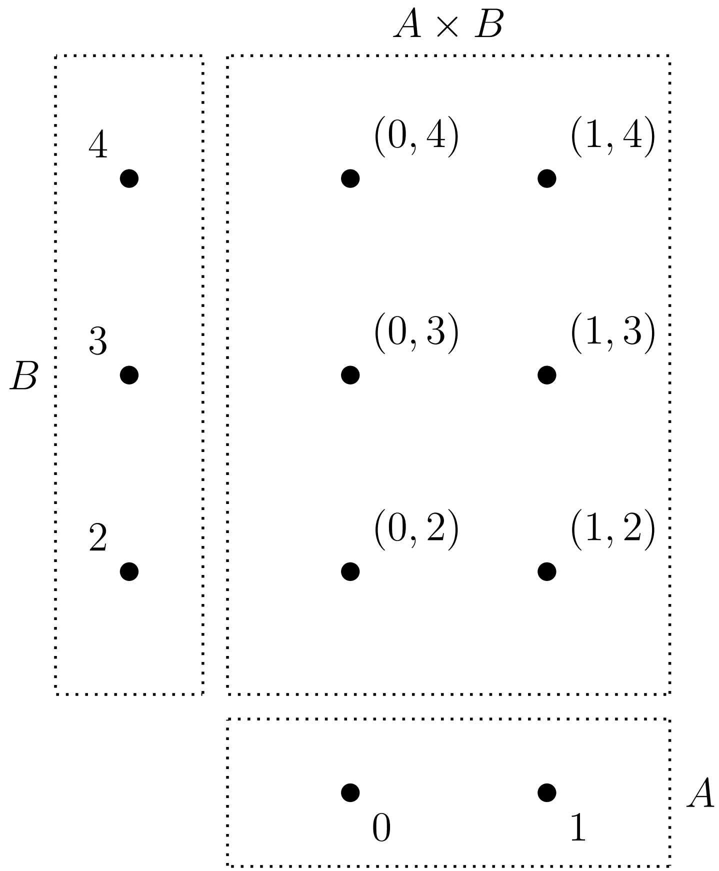
Remarks.

(1) 'Short-hand': $A \times B = \{(x, y) \mid x \in A \text{ and } y \in B\}$.

(2) When $A = B$, we write $A \times B$ as A^2.

Examples.

(a) $\{0,1\} \times \{2,3,4\} = \{(0,2),(0,3),(0,4),(1,2),(1,3),(1,4)\}$.



(b) $\mathbb{R} \times \mathbb{R}$ is the 'coordinate plane' $\mathbb{R}^2$ in school mathematics.

4. **Ordered pairs as set-theoretic objects**.

Philosphical question. *How to 'make sense' of the notion of ordered pairs in set language, in terms of objects already introduced in set language?*

Definition.

Let x, y be objects.

The **Kuratowski ordered pair** *of x, y, with x being the first coordinate and y being the second coordinate, is defined to be the set*

$$\{ \{x\}, \{x, y\} \},$$

How to form this object from the objects x, y?
① Start with x, y.
② Form the sets $\{x\}, \{x, y\}$.
Regard them as some objects.
③ Now form the set $\{ \{x\}, \{x, y\} \}$.

and is denoted by $(x, y)_K$.

This definition is appropriate because we have:

Lemma (OP).

Suppose x, y, u, v are objects. Then $(x, y)_K = (u, v)_K$ iff ($x = u$ and $y = v$).

Proof. Exercise in set language (playing around with '$\ldots \in \{\ldots\}$' and logic).

Remark. From now on, we write $(x, y)_K$ as (x, y).

Further remark. Another version of definition for the notion of ordered pairs?

Wiener's version: $(x, y)_W = \{ \{\emptyset, \{x\}\}, \{\{y\}\} \}$.

5. **Ordered triples and beyond**.

We define the notion for ordered triples in terms of ordered pairs.

Definition.

Let x, y, z be objects. We define the **ordered triple** *of x, y, z, with first, second, third coordinates being x, y, z respectively, to be $((x, y), z)$. We write it as (x, y, z).*

This definition is appropriate because of the validity of Lemma (OT).

Lemma (OT).

Suppose x, y, z, u, v, w are objects. Then $(x, y, z) = (u, v, w)$ iff ($x = u$ and $y = v$ and $z = w$).

Proof. Exercise.

Remark.

We may extend the idea in the definition for the notion of ordered triple so as to give the definition for the notions of ordered quadruples, ordered quintuples et cetera.

6. **Theorem ($*$). (Set-theoretic properties of cartesian products.)**

Let A, B, C, D be sets. The following statements hold:

(1) $(A \cap B) \times (C \cap D) = (A \times C) \cap (B \times D)$. *Also,*

$$(A \cap B) \times (C \cap D) = (A \times D) \cap (B \times C).$$

(2) $(A \cup B) \times C = (A \times C) \cup (B \times C)$. *Also,*
$A \times (C \cup D) = (A \times C) \cup (A \times D)$.

(3) *Suppose* $A \subset B$ *and* $C \subset D$. *Then* $A \times C \subset B \times D$.

(4) *Suppose* $A \neq \emptyset$, $A \subset B$ *and* $A \times C \subset B \times D$. *Then* $C \subset D$.

(5) $A \times \emptyset = \emptyset$, *and* $\emptyset \times A = \emptyset$.