1. Recall that to dis-prove a statement is the same as to prove the negation of the statement concerned. Equivalently we may prove that the statement concerned is false.

   Here we focus on dis-proofs against statements starting with the existential quantifier. Such dis-proofs are referred to as wholesale refutations (against the statements concerned).

2. **Wholesale refutations.**

Consider a statement of the form below:

N: '*There exists some so-and-so amongst the elements of the set blah-blah-blah such that so-and-so satisfies bleh-bleh-bleh.*'

And also consider its variation below:

N_1: '*There exists some so-and-so amongst such that so-and-so is amongst elements of the set blah-blah-blah and so-and-so satisfies bleh-bleh-bleh.*'

N is a statement starting with an existential quantifier:

$$(\exists x)(J(x) \wedge L(x)).$$

$J(x)$ corresponds to the part '*so-and-so is an element of the set blah-blah-blah*'. $L(x)$ corresponds to the part '*so-and-so satisfies bleh-bleh-bleh*'.

To dis-prove the statement N, we may choose either of these two strategies:

(A) prove the statement $\sim N$.

(B) prove a statement of the form $N \longrightarrow C$, in which C is a known contradiction.

Both are equally ‘legitimate’ from the point of view of logic.

Depending on the ‘concrete’ situation, one strategy may be easier to implement than the other.

Whichever strategy is chosen, the argument is called **wholesale refutation** against N.

We consider these two strategies separately.

3. **Wholesale refutation through proving the negation of the statement to be dis-proved.**

The negation $\sim N$ of the statement N is a statement starting with a universal quantifier:

$$(\forall x)((\sim J(x)) \vee (\sim L(x))).$$

As $J(x)$ refers to the part 'so-and-so is an element of the set *blah-blah-blah*', it is sometimes more convenient to write $\sim N$ as:

$$(\forall x)(J(x) \rightarrow (\sim L(x))).$$

In a 'wordy' form, it reads:

$\sim N$: '*For any so-and-so, if so-and-so is amongst the elements of the set blah-blah-blah then so-and-so does not satisfy bleh-bleh-bleh.*'

Hence to prove $\sim N$, we argue that every so-and-so amongst the elements of the set *blah-blah-blah* fails to satisfy *bleh-bleh-bleh*.

When we dis-prove N by directly proving $\sim N$, we proceed as described below:

(A) Pick any object so-and-so. (Throughout the rest of the argument this so-and-so is fixed.) Then we choose any one of the three approaches below:

(A1) Suppose this so-and-so is amongst the elements of the set *blah-blah-blah*.
Then deduce that this so-and-so fails to satisfy *bleh-bleh-bleh*.

(A2) Suppose this so-and-so satisfies *bleh-bleh-bleh*.
Then deduce that this so-and-so is not amongst the elements of the set *blah-blah-blah*.

(A3) Suppose this so-and-so is amongst the elements of the set *blah-blah-blah* and also suppose this so-and-so satisfy *bleh-bleh-bleh*.
Then look for a contradiction.

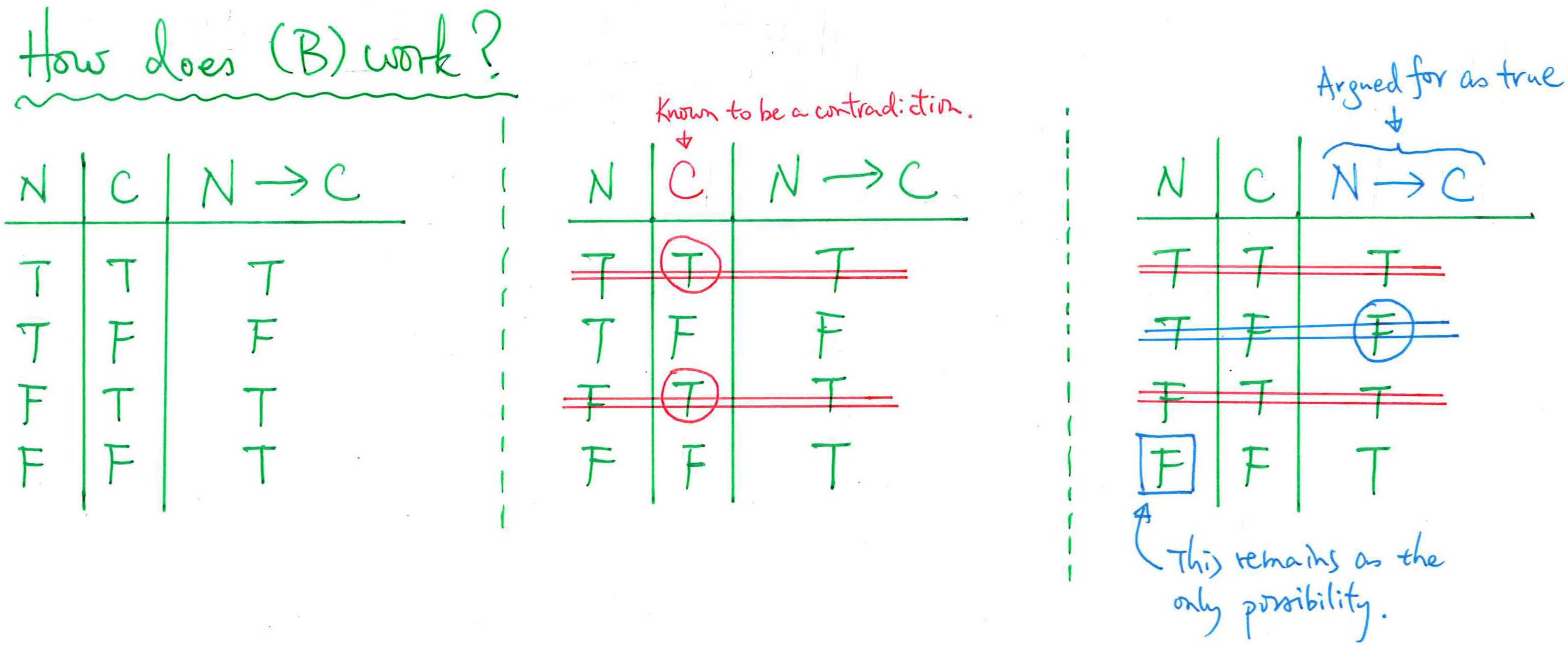
4. **Wholesale refutation through obtaining a contradiction from the statement to be dis-proved.**

When we dis-prove N by proving a statement of the form $N \longrightarrow C$, in which C is a known contradiction, we proceed as described below:

(B) Suppose it were true that there existed some so-and-so amongst the elements of the set *blah-blah-blah* such that so-and-so satisfied *bleh-bleh-bleh*.

Then look for a contradiction that arises from the existence of such a so-and-so.

This may be no easy task because the desired contradiction is not specified in the first place.



5. **Generalization.**

More generally, to dis-prove a statement of the form

$$\underbrace{(\exists x)(\exists y)\cdots(\exists z)}_{\text{all }\exists\text{'s}}(J(x,y,\cdots,z)\wedge L(x,y,\cdots,z)),$$

we may also choose either of these two strategies:

(A) prove its negation, which is the statement

$$\underbrace{(\forall x)(\forall y)\cdots(\forall z)}_{\text{all }\forall\text{'s}}[(\sim J(x,y,\cdots,z))\vee(\sim L(x,y,\cdots,z)],$$

in any one of its formulation.

(B) prove a statement of the form

$$[\underbrace{(\exists x)(\exists y)\cdots(\exists z)}_{\text{all }\exists\text{'s}}(J(x,y,\cdots,z)\wedge L(x,y,\cdots,z))]\longrightarrow C,$$

in which C is a known contradiction.

6. **Examples of 'wholesale refutations'.**

(a) We want to dis-prove

N: *The equation $x^2 + 1 = 0$ has some real solution.*

This statement is actually an existence statement in disguise:

There exists some object p such that $p \in \mathbb{R}$ and $p^2 + 1 = 0$.

(A) We may dis-prove N by proving $\sim N$ in this formulation:

- For any object p, if $p \in \mathbb{R}$ then $p^2 + 1 \neq 0$.

Hence we write:

- Suppose $p \in \mathbb{R}$. [Try to deduce: $p^2 + 1 \neq 0$.]
  We have $p^2 \geq 0$. Then $p^2 + 1 \geq 1 > 0$. Therefore $p^2 + 1 \neq 0$. □

(B) We dis-prove N by obtaining a contradiction from the assumption of N. Hence we write:

- Suppose it were true that there existed some $p \in \mathbb{R}$ such that $p^2 + 1 = 0$.
  Then, since $p \in \mathbb{R}$, we would have $p^2 \geq 0$.
  Therefore $0 = p^2 + 1 \geq 1 > 0$. Contradiction arises.
  (Hence the statement 'there exists some $p \in \mathbb{R}$ such that $p^2 + 1 = 0$' is false.) □

(b) We want to dis-prove

N: There exist some $z \in \mathbb{C}$ *such that* $|z| > |\mathsf{Re}(z)| + |\mathsf{Im}(z)|$.

(A) We may dis-prove N by proving $\sim N$ in this formulation:

- For any $z \in \mathbb{C}$, the inequality $|z| \le |Re(z)| + |Im(z)|$ holds

Hence we write:

- Pick any $z \in \mathbb{C}$.

$$|z|^2 = (Re(z))^2 + (Im(z))^2 \le |Re(z)|^2 + |Im(z)|^2 + 2|Re(z)| \cdot |Im(z)| = (|Re(z)| + |Im(z)|)^2.$$

Then, since $|z| \ge 0$ and $|Re(z)| + |Im(z)| \ge 0$, we have $|z| \le |Re(z)| + |Im(z)|$. □

(B) We may dis-prove N by obtaining a contradiction under the assumption N. Hence we write:

- Suppose it were true that there existed some $z \in \mathbb{C}$ such that $|z| > |Re(z)| + |Im(z)|$.

Then, for this z, we would have

$$(Re(z))^2 + (Im(z))^2 = |z|^2 > (|Re(z)| + |Im(z)|)^2 = |Re(z)|^2 + |Im(z)|^2 + 2|Re(z)| \cdot |Im(z)|$$

← [Why?]

Therefore $0 > 2|Re(z)| \cdot |Im(z)| \ge 0$.

Contradiction arises. □

(c) We want to dis-prove

N: *The set* $[0, 1)$ *has a greatest element.*

This statement is actually an existence statement in disguise:

There exists some $\lambda \in [0,1)$ such that (for any $x \in [0,1)$, $x \leq \lambda$).

(A) We may dis-prove N by proving $\sim N$ in this formulation:

- For any $\lambda \in [0,1)$, there exists some $x_0 \in [0,1)$ such that $x_0 > \lambda$.

Hence we write:

- Pick any $\lambda \in [0,1)$.
  $0 \leq \lambda < 1$.
  Take $x_0 = \frac{\lambda+1}{2}$. Then $0 \leq \lambda < x_0 < 1$.
  Therefore $x_0 \in [0,1)$ and $x_0 > \lambda$. □

(B) We dis-prove N by obtaining a contradiction under the assumption N. Hence we write:

- Suppose it were true that the set $[0,1)$ had a greatest element, say, λ.
  By definition, $\lambda \in [0,1)$. Then $0 \leq \lambda < 1$.
  Take $x_0 = \frac{\lambda+1}{2}$. Then $0 \leq \lambda < x_0 < 1$. Therefore $x_0 \in [0,1)$ and $x_0 > \lambda$.
  But λ was a greatest element of $[0,1)$. Contradiction arises. □

(d) We want to dis-prove

N: *Some circle on the Argand plane passes through all four points* $1, -1, i, 1+i$.

Very formally presented, the statement *N* is:

There exists some subset T of $\mathbb{C}$ such that
T is a circle and ($1 \in T$ and $-1 \in T$ and $i \in T$ and $1+i \in T$)

(A) We may dis-prove *N* by proving $\sim N$. Hence we write:

- [Try to prove: For any subset T of $\mathbb{C}$, if T is a circle then at least one of $1, -1, i, 1+i$ does not belong to T.]

... ← [Your work.]

(B) We dis-prove *N* by obtaining a contradiction from the assumption of *N*. Hence we write:

- Suppose it were true that there existed some subset T of $\mathbb{C}$ such that T was a circle and $1 \in T$ and $-1 \in T$ and $i \in T$ and $1+i \in T$.

The equation of T is given by $|z-\alpha| = r$, for some appropriate $\alpha \in \mathbb{C}$, $r > 0$.

Since $1, -1, i \in T$, we would have $\alpha = 0$ and $r = 1$. [Fill in the detail.]

Note that $|1+i| = \sqrt{2} \neq 1$. Then $1+i \notin T$. Contradiction arises. □

(e) We want to dis-prove

> N: *There exists some non-zero* (3×3)*-square matrix* P *such that* P *is symmetric and* P *is skew-symmetric.*

Very formally presented, the statement N is:

> *There exists some* (3×3)*-square matrix* P *such that* P *is non-zero and* P *is symmetric and* P *is skew-symmetric.*

(A) We may dis-prove N by proving $\sim N$ directly.

Note that $\sim N$ can be formulated as:

> *For any* (3×3)*-square matrix* P*, if* P *is symmetric and* P *is skew-symmetric then* $P = 0$.

Hence we write:

- Let P be a (3×3)-square matrix P.
  Suppose P is symmetric and P is skew-symmetric.
  Since P is symmetric, we have $P^t = P$.
  Also, since P is skew-symmetric, we have $P^t = -P$.
  Then $2P = P + P = P^t + (-P^t) = 0$. Therefore $P = 0$.

We want to dis-prove

N: There exists some non-zero (3×3)-square matrix P such that P is symmetric and P is skew-symmetric.

Very formally presented, the statement N is:

There exists some (3×3)-square matrix P such that P is non-zero and P is symmetric and P is skew-symmetric.

(B) We dis-prove N by obtaining a contradiction under the assumption N. Hence we write:

- Suppose there existed some (3×3)-square matrix P such that P was non-zero and P was symmetric and P was skew-symmetric.
  By assumption, since P was symmetric, $P^t = P$.
  Also, since P was skew-symmetric, $P^t = -P$.
  Then $2P = P + P = P^t + (-P^t) = 0$. Therefore $P = 0$.
  But P was non-zero by definition. Contradiction arises.
  Hence it is false in the first place that there existed some (3×3)-square matrix P such that P was non-zero and P was symmetric and P was skew-symmetric.

7. **Warning on common mistakes.**

'$\sim((\exists x)P(x))$', '$(\exists x)(\sim P(x))$' are different statements.

'$\sim((\exists x \in S)Q(x))$', '$(\exists x \in S)(\sim Q(x))$' are different statements.

If you attempt to dis-prove a statement of the form

> '*there exists some x such that ($P(x)$ holds)*' (or '*there exists some $x \in S$ such that ($Q(x)$ holds)*' respectively)

by proceeding to prove the statement

> '*there exists some x such that ($P(x)$ does not hold)*' (or '*there exists some $x \in S$ such that ($Q(x)$ does not hold)*')

you will end up achieving too little of value.