MATH1050 Dis-proofs by counter-example.

1. **What is 'dis-proving'?**

   To dis-prove a statement is the same as to prove the negation of the statement concerned. Equivalently we may prove that the statement concerned is false.

   To proceed to give a dis-proof for a statement, we have to first distinguish whether it starts with a universal quantifier or with a existential quantifier. For the former, we give a **dis-proof by counter-example**. As for the latter, we give a '**wholesale refutation**'.

   Here we will be concerned with dis-proofs by counter-example. For wholesale refutation, refer to the handout *dis-proofs by wholesale refutation.*

2. **Dis-proofs by counter-example.**

   Consider a statement of the form below:

   M: '*Let so-and-so be an element of the set blah-blah-blah. Suppose so-and-so satisfies bleh-bleh-bleh. Then so-and-so satisfies bloh-bloh-bloh.*'

   And also consider its variation in the forms below:

   M_1: '*Let so-and-so be an element of the set blah-blah-blah. So-and-so satisfies bloh-bloh-bloh.*'

   M_2: '*Suppose so-and-so satisfies bleh-bleh-bleh. Then so-and-so satisfies bloh-bloh-bloh.*'

   M is actually a statement starting with a universal quantifier:

   $$(\forall x)(H(x) \to K(x)).$$

   $H(x)$ corresponds to the part '*so-and-so is an element of the set blah-blah-blah and so-and-so satisfies bleh-bleh-bleh*'. $K(x)$ corresponds to the part '*so-and-so satisfies bloh-bloh-bloh*'.

   The negation $\sim M$ of the statement M is a statement starting with an existential quantifier:

   $$(\exists x)(H(x) \wedge (\sim K(x))).$$

   In a 'wordy' form, it reads:

   $\sim M$: '*There exists so-and-so amongst the elements of the set blah-blah-blah such that so-and-so satisfies bleh-bleh-bleh and so-and-so does not satisfy bloh-bloh-bloh.*'

   Therefore to prove $\sim M$, we proceed by naming one 'concrete' so-and-so amongst the elements of the set *blah-blah-blah* which indeed satisfies *bleh-bleh-bleh* and which does not satisfy *bloh-bloh-bloh*. Such a concrete so-and-so is called a **counter-example** for the statement M which we dis-prove.

   More generally, to dis-prove a statement of the form

   $$\underbrace{(\forall x)(\forall y)\cdots(\forall z)}_{\text{all } \forall\text{'s}}(H(x, y, \cdots, z) \longrightarrow K(x, y, \cdots, z)),$$

   we prove its negation, which is the statement

   $$\underbrace{(\exists x)(\exists y)\cdots(\exists z)}_{\text{all } \exists\text{'s}}[H(x, y, \cdots, z) \wedge (\sim K(x, y, \cdots, z)].$$

   We refer to such an argument a dis-proof-by-counter-example.

3. **Simple examples of dis-proofs-by-counter-example.**

   (a) We want to dis-prove

   M: *Let* $x \in \mathbb{R}$. $x^2 > 0$.

   Very formally presented, M is:

   For any object x, *(if* $x \in \mathbb{R}$ *then* $x^2 > 0$*).*

   So the statement $\sim M$ reads:

There exists some object x_0 such that ($x_0 \in \mathbb{R}$ and ${x_0}^2 \leq 0$).

A dis-proof by counter-example against the statement M is:

- Take $x_0 = 0$.
  Note that $x_0 \in \mathbb{R}$.
  Also ${x_0}^2 = 0^2 = 0 \leq 0$.
  So the statement '${x_0}^2 > 0$' is false.

(b) We want to dis-prove

M: Let $n \in \mathbb{N}$. Suppose n is divisible by 3. Then n is divisible by 5.

Very formally presented, M is:

For any object n, if $n \in \mathbb{N}$ and n is divisible by 3 then n is divisible by 5.

So $\sim M$ reads:

There exists some object n_0 such that ($n_0 \in \mathbb{N}$ and n_0 is divisible by 3 and n is not divisible by 5).

A dis-proof by counter-example for the statement M is:

- Take $n_0 = 3$.
  Note that $n_0 \in \mathbb{N}$.
  Also, n_0 is divisible by 3. (Detail?)
  But n_0 is not divisible by 5. (Detail?)

4. **Further examples of dis-proofs-by-counter-example.**

(a) We want to dis-prove

M: Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x. Then $x = y$.

Very formally presented, M is:

For any objects x, y, [if ($x \in \mathbb{Z}$, $y \in \mathbb{Z}$ and x is divisible by y and y is divisible by x) then $x = y$].

So $\sim M$ reads:

There exist some objects x, y such that ($x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ and x is divisible by y and y is divisible by x) and $x \neq y$].

A dis-proof by counter-example against the statement M is:

- Take $x_0 = 1$, $y_0 = -1$.
  Note that $x_0, y_0 \in \mathbb{Z}$.
  Also, x_0 is divisible by y_0 and y_0 is divisible by x_0. (Detail?)
  But $x_0 \neq y_0$.

(b) We want to dis-prove

M: Let a, b be rational numbers. $a + b\sqrt{2}$ is irrational.

Very formally presented, M is:

For any objects a, b, [if (a is a rational number and b is a rational number) then $a + b\sqrt{2}$ is irrational].

So $\sim M$ reads:

There exist some objects a, b such that (a is a rational number and b is a rational number and $a + b\sqrt{2}$ is not irrational).

A dis-proof by counter-example against the statement M is:

- Take $a_0 = 0$, $b_0 = 0$. a_0, b_0 are rational numbers. $a_0 + b_0\sqrt{2} = 0$. $a_0 + b_0\sqrt{2}$ is not irrational.

(c) We want to dis-prove

M: Let $r, s, t \in \mathbb{R}$. Suppose r is a non-zero rational number and s is an irrational number. Then both $rs + t, rs - t$ are irrational numbers.

A dis-proof by counter-example against the statement M is:

- Take $r_0 = 1$, $s_0 = \sqrt{2}$, $t_0 = \sqrt{2}$. Note that r_0, s_0, t_0 are real numbers.
  r_0 is a non-zero rational number.
  s_0 is an irrational number.
  Note that $r_0 s_0 - t_0 = 0$.
  One of $r_0 s_0 + t_0, r_0 s_0 - t_0$, namely the latter, is not irrational.

(d) We want to dis-prove

M: The product of any two distinct irrational numbers is irrational.

At first sight this statement is not of the form 'if blah-blah-blah then bleh-bleh-bleh'. However, M can be re-written in this way:

Let u, v be irrational numbers. Suppose $u \neq v$. Then uv is an irrational number.

A dis-proof by counter-example against the statement M is:

- Take $u_0 = \sqrt{2}$, $v_0 = -\sqrt{2}$.
  Note that u_0, v_0 are irrational numbers, and $u_0 \neq v_0$.
  Note that $u_0 v_0 = -2$. Then $u_0 v_0$ is not an irrational number.

(e) We want to dis-prove

M: Let A, B, C be sets. Suppose $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$ and $C \cap A \neq \emptyset$. Then $A \cap B \cap C \neq \emptyset$.

Very formally presented, M is:

For any sets A, B, C, if ($A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$ and $C \cap A \neq \emptyset$) then $A \cap B \cap C = \emptyset$.

So $\sim M$ reads:

There exist some sets A_0, B_0, C_0 such that ($A_0 \cap B_0 \neq \emptyset$ and $B_0 \cap C_0 \neq \emptyset$ and $C_0 \cap A_0 \neq \emptyset$ and $A_0 \cap B_0 \cap C_0 = \emptyset$).

A dis-proof by counter-example against the statement M is:

- Take $A_0 = \{0, 1\}$, $B_0 = \{1, 2\}$, $C_0 = \{0, 2\}$. Here $0, 1, 2$ are regarded as pairwise distinct objects.
  Note that $A_0 \cap B_0 = \{1\}$, $B_0 \cap C_0 = \{2\}$, $C_0 \cap A_0 = \{0\}$.
  So $A_0 \cap B_0 \neq \emptyset$ and $B_0 \cap C_0 \neq \emptyset$ and $C_0 \cap A_0 \neq \emptyset$. (Detail?)
  Note that $A_0 \cap B_0 \cap C_0 = \emptyset$.

(f) We want to dis-prove

M: Let n be a positive integer, and P, Q be $(n \times n)$-square matrices with real entries. Suppose $PQ = 0$. Then $P = 0$ or $Q = 0$.

(Here 0 stands for the zero (2×2)-square matrix.)

Very formally presented, M is:

For any positive integer n, for any $(n \times n)$-square matrices P, Q with real entries, if $PQ = 0$ then ($P = 0$ or $Q = 0$).

So $\sim M$ reads:

There exist some positive integer n, some $(n \times n)$-square matrices P, Q with real entries such that $PQ = 0$ and ($P \neq 0$ and $Q \neq 0$).

A dis-proof by counter-example against the statement M is:

- Take $n = 2$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We have $P \neq 0$ and $Q \neq 0$.

  Note that $PQ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$.

5. **Warning on common mistakes.**

(a) '$\sim((\forall x)P(x))$', '$(\forall x)(\sim P(x))$' are different statements.
'$\sim((\forall x \in S)Q(x))$', '$(\forall x \in S)(\sim Q(x))$' are different statements.
If you try to dis-prove a statement of the form

'*for any x, ($P(x)$ holds)*' (or '*for any $x \in S$, ($Q(x)$ holds)*' respectively)

by proceeding to prove the statement

'*for any x, ($P(x)$ does not hold)*' (or '*for any $x \in S$, ($Q(x)$ does not hold)*' respectively)

you will probably be attempting to achieve the impossible and end up nowhere.

(b) '$\sim((\forall x)(H(x) \to K(x))$', '$(\forall x)[H(x) \to (\sim K(x))]$' are different statements.
If you try to dis-prove a statement of the form

'*for any x, (if $H(x)$ holds then $K(x)$ holds)*'

by proceeding to prove the statement

'*for any x, (if $H(x)$ holds then $K(x)$ does not hold)*'

you will probably be attempting to achieve the impossible and end up nowhere.