1. **What is 'dis-proving'?**

   To dis-prove a statement is the same as to prove the negation of the statement concerned. Equivalently we may prove that the statement concerned is false.

   To proceed to give a dis-proof for a statement, we have to first distinguish whether it starts with a universal quantifier or with a existential quantifier.

   - To dis-prove a statement starting with the universal quantifier, we give a **dis-proof by counter-example**.
   - To dis-prove a statement starting with the existential quantifier, we give a '**wholesale refutation**'.

   Here we will be concerned with dis-proofs by counter-example.

2. **Dis-proofs by counter-example.**

Consider a statement of the form below:

M: '*Let so-and-so be an element of the set blah-blah-blah. Suppose so-and-so satisfies bleh-bleh-bleh. Then so-and-so satisfies bloh-bloh-bloh.*'

And also consider its variation in the forms below:

M_1: '*Let so-and-so be an element of the set blah-blah-blah. So-and-so satisfies bloh-bloh-bloh.*'

M_2: '*Suppose so-and-so satisfies bleh-bleh-bleh. Then so-and-so satisfies bloh-bloh-bloh.*'

M is actually a statement starting with a universal quantifier:

$$(\forall x)(H(x) \to K(x)).$$

$H(x)$ corresponds to the part '*so-and-so is an element of the set blah-blah-blah and so-and-so satisfies bleh-bleh-bleh*'.

$K(x)$ corresponds to the part '*so-and-so satisfies bloh-bloh-bloh*'.

The negation $\sim M$ of the statement M is a statement starting with an existential quantifier:

$$(\exists x)(H(x) \wedge (\sim K(x))).$$

In a 'wordy' form, it reads:

$\sim M$: '*There exists so-and-so amongst the elements of the set **blah-blah-blah** such that so-and-so satisfies **bleh-bleh-bleh** and so-and-so does not satisfy **bloh-bloh-bloh**.*'

Therefore to prove $\sim M$, we proceed by naming one 'concrete' so-and-so amongst the elements of the set *blah-blah-blah* which indeed satisfies *bleh-bleh-bleh* and which does not satisfy *bloh-bloh-bloh*.

Such a concrete so-and-so is called a **counter-example** for the statement M which we dis-prove.

More generally, to dis-prove a statement of the form

$$\underbrace{(\forall x)(\forall y)\cdots(\forall z)}_{\text{all }\forall\text{'s}}(H(x,y,\cdots,z) \longrightarrow K(x,y,\cdots,z)),$$

we prove its negation, which is the statement

$$\underbrace{(\exists x)(\exists y)\cdots(\exists z)}_{\text{all }\exists\text{'s}}[H(x,y,\cdots,z) \wedge (\sim K(x,y,\cdots,z)].$$

We refer to such an argument a dis-proof-by-counter-example.

3. **Simple examples of dis-proofs-by-counter-example.**

(a) We want to dis-prove

M: Let $x \in \mathbb{R}$. $x^2 > 0$.

Very formally presented, M is:

For any object x, (if $x \in \mathbb{R}$ then $x^2 > 0$).

So the statement $\sim M$ reads:

There exists some object x_0 such that ($x_0 \in \mathbb{R}$ and ${x_0}^2 \leq 0$).

A dis-proof by counter-example against the statement M is:

- Take $x_0 = 0$.

  $x_0 \in \mathbb{R}$.

  $x_0^2 = 0^2 = 0 \leq 0$.

  (Hence the statement '$\text{Let } x \in \mathbb{R}.\ x^2 > 0.$' is false.) $\square$

(b) We want to dis-prove

M: Let $n \in \mathbb{N}$. Suppose n is divisible by 3. Then n is divisible by 5.

Very formally presented, M is:

For any object n, if $n \in \mathbb{N}$ and n is divisible by 3 then n is divisible by 5.

So $\sim M$ reads:

There exists some object n_0 such that ($n_0 \in \mathbb{N}$ and n_0 is divisible by 3 and n_0 is not divisible by 5).

A dis-proof by counter-example against the statement M is:

- Take $n_0 = 3$.
  $n_0 \in \mathbb{N}$.
  n_0 is divisible by 3. (Reason: $n_0 = 3 \cdot 1$ and $1 \in \mathbb{Z}$.)
  Claim: n_0 is not divisible by 5.
  Justification of this claim: Suppose it were true that n_0 was divisible by 5.
  Then, by the definition of divisibility, ...
  Contradiction arises. $\square$ [Your work.]

Remark. What is wrong with the 'argument' below? What is wrong with the logic?

- Let $n \in \mathbb{N}$. Suppose n is divisible by 3.
  Take $n = 3$. Then n is divisible by 3. Also, n is not divisible by 5.

4. **Further examples of dis-proofs-by-counter-example.**

(a) We want to dis-prove

M: Let $x, y \in \mathbb{Z}$. Suppose x is divisible by y and y is divisible by x. Then $x = y$.

Very formally presented, M is:

For any objects x, y, [if ($x \in \mathbb{Z}$, $y \in \mathbb{Z}$ and x is divisible by y and y is divisible by x) then $x = y$].

So $\sim M$ reads:

There exist some objects x, y such that ($x \in \mathbb{Z}$ and $y \in \mathbb{Z}$ and x is divisible by y and y is divisible by x) and $x \neq y$].

A dis-proof by counter-example against the statement M is:

- Take $x_0 = 1$, $y_0 = -1$.

  $x_0, y_0 \in \mathbb{Z}$.

  x_0 is divisible by y_0 and y_0 is divisible by x_0. [Fill in the reason.]

  $x_0 \neq y_0$. $\square$

(b) We want to dis-prove

M: *Let* a, b *be rational numbers.* $a + b\sqrt{2}$ *is irrational.*

Very formally presented, M is:

For any objects a, b, *[if (*a *is a rational number and* b *is a rational number) then* $a + b\sqrt{2}$ *is irrational].*

So $\sim M$ reads:

There exist some objects a, b *such that (*a *is a rational number and* b *is a rational number and* $a + b\sqrt{2}$ *is not irrational).*

A dis-proof by counter-example against the statement M is:

- Take $a_0 = 0$, $b_0 = 0$.

  $a_0, b_0 \in \mathbb{Q}$.

  $a_0 + b_0\sqrt{2} = 0 \in \mathbb{Q}$. $\square$

(c) We want to dis-prove

> M: *Let $r, s, t \in \mathbb{R}$. Suppose r is a non-zero rational number and s is an irrational number. Then both $rs + t, rs - t$ are irrational numbers.*

A dis-proof by counter-example against the statement M is:

- [M reads: For any $r, s, t \in \mathbb{R}$, if $(r \in \mathbb{Q} \setminus \{0\}$ and $s \in \mathbb{R} \setminus \mathbb{Q})$ then $(rs+t \in \mathbb{R} \setminus \mathbb{Q}$ and $rs-t \in \mathbb{R} \setminus \mathbb{Q})$.
  $\sim M$ reads: There exist some $r_0, s_0, t_0 \in \mathbb{R}$ such that $r_0 \in \mathbb{Q} \setminus \{0\}$ and $s_0 \in \mathbb{R} \setminus \mathbb{Q}$ and $(r_0 s_0 + t_0 \in \mathbb{Q}$ or $r_0 s_0 - t_0 \in \mathbb{Q})$.]

  Take $r_0 = 1$, $s_0 = \sqrt{2}$, $t_0 = \sqrt{2}$.

  $r_0, s_0, t_0 \in \mathbb{R}$.

  $r_0 \in \mathbb{Q} \setminus \{0\}$. $s_0 \in \mathbb{R} \setminus \mathbb{Q}$.

  $r_0 s_0 - t_0 = 0 \in \mathbb{Q}$. Then $(r_0 s_0 + t_0 \in \mathbb{Q}$ or $r_0 s_0 - t_0 \in \mathbb{Q})$. $\square$

Remark. What is wrong with the 'argument' below for $\sim M$? What is wrong with the logic?

- Let $r, s, t \in \mathbb{R}$. Suppose r is a non-zero rational number and s is an irrational number.

  [Try to deduce: $rs+t \in \mathbb{Q}$ or $rs-t \in \mathbb{Q}$.]

  Suppose $rs+t \in \mathbb{R} \setminus \mathbb{Q}$. [Try to deduce: $rs-t \in \mathbb{Q}$.] ...

(d) We want to dis-prove

M: *The product of any two distinct irrational numbers is irrational.*

At first sight this statement is not of the form 'if blah-blah-blah then bleh-bleh-bleh'. However, M can be re-written in this way:

Let u, v be irrational numbers. Suppose $u \neq v$. Then uv is an irrational number.

A dis-proof by counter-example against the statement M is:

- [$\sim M$ reads: There exist some $x_0, y_0 \in \mathbb{R}$ such that ($x_0 \neq y_0$ and $x_0 \in \mathbb{R} \setminus \mathbb{Q}$ and $y_0 \in \mathbb{R} \setminus \mathbb{Q}$) and $x_0 y_0 \in \mathbb{Q}$.]

  Take $x_0 = \sqrt{2}$, $y_0 = 2\sqrt{2}$.

  $x_0, y_0 \in \mathbb{R}$.

  $x_0 \neq y_0$.

  x_0, y_0 are both irrational. [Fill in the reason.]

  $x_0 y_0 = (\sqrt{2}) \cdot (2\sqrt{2}) = 4 \in \mathbb{Q}$. Then $x_0 y_0$ is not irrational. $\square$

Remark. What is wrong in trying to prove 'The product of any two distinct irrational number is rational'? What is wrong with the logic?

(e) We want to dis-prove

> *M: Let A, B, C be sets. Suppose $A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$ and $C \cap A \neq \emptyset$. Then $A \cap B \cap C \neq \emptyset$.*

Very formally presented, M is:

> *For any sets A, B, C, if ($A \cap B \neq \emptyset$ and $B \cap C \neq \emptyset$ and $C \cap A \neq \emptyset$) then $A \cap B \cap C = \emptyset$.*

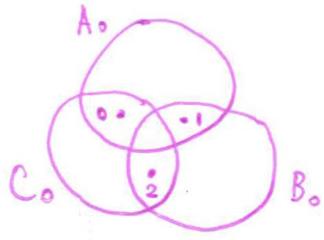
So $\sim M$ reads:

> *There exist some sets A_0, B_0, C_0 such that ($A_0 \cap B_0 \neq \emptyset$ and $B_0 \cap C_0 \neq \emptyset$ and $C_0 \cap A_0 \neq \emptyset$ and $A_0 \cap B_0 \cap C_0 = \emptyset$).*

A dis-proof by counter-example against the statement M is:

- Take $A_0 = \{0, 1\}$, $B_0 = \{1, 2\}$, $C_0 = \{2, 0\}$.
  Here $0, 1, 2$ are regarded as pairwise distinct objects.
  Note that $A_0 \cap B_0 = \{1\}$, $B_0 \cap C_0 = \{2\}$, $C_0 \cap A_0 = \{0\}$.
  So $A_0 \cap B_0 \neq \phi$ and $B_0 \cap C_0 \neq \phi$ and $C_0 \cap A_0 \neq \phi$.
  Note that $A_0 \cap B_0 \cap C_0 = \phi$. $\square$

Roughwork.



Make things as simple as possible.

(f) We want to dis-prove

M: Let n be a positive integer, and P, Q be $(n \times n)$-square matrices with real entries. Suppose $PQ = 0$. Then $P = 0$ or $Q = 0$.

(Here 0 stands for the zero (2×2)-square matrix.)

Very formally presented, M is:

For any positive integer n, for any $(n \times n)$-square matrices P, Q with real entries, if $PQ = 0$ then ($P = 0$ or $Q = 0$).

So $\sim M$ reads:

There exist some positive integer n, some $(n \times n)$-square matrices P, Q with real entries such that $PQ = 0$ and ($P \neq 0$ and $Q \neq 0$).

A dis-proof by counter-example against the statement M is:

- Take $n = 2$, $P = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, $Q = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. We have $P \neq 0$ and $Q \neq 0$.

  Note that $PQ = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$.

5. **Warning on common mistakes.**

(a) '$\sim((\forall x)P(x))$', '$(\forall x)(\sim P(x))$' are different statements.

'$\sim((\forall x \in S)Q(x))$', '$(\forall x \in S)(\sim Q(x))$' are different statements.

If you try to dis-prove a statement of the form

'*for any* x, *(*$P(x)$ *holds)*' (or '*for any* $x \in S$, *(*$Q(x)$ *holds)*' respectively)

by proceeding to prove the statement

'*for any* x, *(*$P(x)$ *does not hold)*' (or '*for any* $x \in S$, *(*$Q(x)$ *does not hold)*' respectively)

you will probably be attempting to achieve the impossible and end up nowhere.

(b) '$\sim((\forall x)(H(x) \rightarrow K(x))$', '$(\forall x)[H(x) \rightarrow (\sim K(x))]$' are different statements.

If you try to dis-prove a statement of the form

'*for any* x, *(if* $H(x)$ *holds then* $K(x)$ *holds)*'

by proceeding to prove the statement

'*for any* x, *(if* $H(x)$ *holds then* $K(x)$ *does not hold)*'

you will probably be attempting to achieve the impossible and end up nowhere.