1. **Predicates**.

   A **predicate with variables** $x, y, z, \cdots$ is a statement 'modulo' the ambiguity of possibly one or several variables $x, y, z, \cdots$. In general, it may fail to be a statement. However, provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

   Statements are predicates with no variable, in light of the above.

   Given predicates $P(x, \cdots)$, $Q(x, \cdots)$, ..., we can make use of the 'logical connectives' to form 'compound' predicates

   - $\sim P(x, \cdots)$,
   - $P(x, \cdots) \wedge Q(x, \cdots)$,
   - $P(x, \cdots) \vee Q(x, \cdots)$,
   - $P(x, \cdots) \rightarrow Q(x, \cdots)$,
   - $P(x, \cdots) \leftrightarrow Q(x, \cdots)$

   et cetera.

2. **Universal quantifier, as the 'generalization' of 'and'**.

   Non-mathematical example.

   - 'Every dog has a tail.'
   - 'For any dog x, x has a tail.'
   - 'For any object x, (if x is a dog then x has a tail).'

   These three sentences mean the same thing. Heuristically, what we mean is 'Dog α has a tail, and Dog β has a tail, and Dog γ has a tail, and ..., and Dog ω has a tail, and ...'. We will very soon be tired with so many 'and'; hence we say 'every dog has a tail'.

   The words '**for any**', '**for all**', '**every**', '**each**', ... indicate the presence of the **universal quantifier**.

   Examples of statements (starting) with one universal quantifier.

   (a) *The square of each real number is non-negative.*

   Formal formulation.

   - *For any $x \in \mathbb{R}$, $x^2 \geq 0$.*

   Very formal formulation.

   - *For any object x, if $x \in \mathbb{R}$ then $x^2 \geq 0$.*

   (b) *Every triangle is equilateral.*

   Formal formulation.

   - *For any triangle T, T is equilateral.*

   Very formal formulation.

   - *For any T, if T is a triangle then T is equilateral.*

   (c) *Let p be a positive prime number. $\sqrt{p}$ is irrational.*

   Formal formulation.

   - *For any positive prime number p, $\sqrt{p}$ is irrational.*

   Very formal formulation.

   - *For any object p, if p is a positive prime number then $\sqrt{p}$ is irrational.*

   (d) *Suppose A is a non-singular square matrix. Then A has an inverse matrix.*

   Formal formulation.

   - *For any non-singular square matrix A, A has an inverse matrix.*

   Very formal formulation.

   - *For any object A, if A is a non-singular square matrix then A has an inverse matrix.*

   A **statement (starting) with one universal quantifier** is a statement of the form

$(\forall x \in S)Q(x)$

(pronounced as 'for all x belonging to S, $Q(x)$'), where S is a given set and $Q(x)$ is a given predicate.

The purist will write '$(\forall x \in S)Q(x)$' as

'$(\forall x)((x \in S) \to Q(x))$'

(pronounced as 'for all x, if x belongs to S then $Q(x)$'). Note that '$x \in S$', $Q(x)$ are predicates, and it makes sense to connect them to form the predicate '$(x \in S) \to Q(x)$'.

3. **Existential quantifier, as the 'generalization' of 'or'.**

Non-mathematical example.

- 'Some dogs have black hair.'
- 'There is at least one dog with black hair.'
- 'There is some dog x so that x has black hair.'
- 'There exists some dog x such that x has black hair.'
- 'There exists some object x such that (x is a dog and x has black hair).'

These five sentences mean the same thing. Heuristically, what we mean is 'Dog α has black hair, or Dog β has black hair, or Dog γ has black hair, or ..., or Dog ω has black hair, or ...'. We will very soon be tired with so many 'or'; hence we say 'some dog has black hair'.

The words '**there exist (some)**', '**there is/are (some)**', '**for some**', '**some**', '**at least one**', '**there is at least one**' indicate the presence of the **existential quantifier**.

Examples of statements (starting) with one existential quantifier.

(a) *There is a prime number.*

Formal formulation.

- *There exists some $x \in \mathbb{Z}$ such that x is a prime number.*

Very formal formulation.

- *There exists some object x such that $x \in \mathbb{Z}$ and x is a prime number.*

(b) *Some non-zero integer is divisible by* 0.

Formal formulation.

- *There exists some $x \in \mathbb{Z}\backslash\{0\}$ such that x is divisible by* 0.

Very formal formulation.

- *There exists some object x such that $x \in \mathbb{Z}\backslash\{0\}$ and x is divisible by* 0.

(c) *Some subset of $\mathbb{R}$ which is bounded above in $\mathbb{R}$ has a greatest element.*

Formal formulation.

- *There exists some subset T of $\mathbb{R}$ such that T is bounded above in $\mathbb{R}$ and T has a greatest element.*

Very formal formulation.

- *There exists some object T such that T is a subset of $\mathbb{R}$ and T is bounded above in $\mathbb{R}$ and T has a greatest element.*

(d) *The equation $\tan(x) = x$ with unknown x in $\mathbb{R}$ has at least one non-zero solution.*

Formal formulation.

- *There exists some $x \in \mathbb{R}\backslash\{0\}$ such that $\tan(x) = x$.*

Very formal formulation.

- *There exists some object x such that $x \in \mathbb{R}\backslash\{0\}$ and $\tan(x) = x$.*

A **statement (starting) with one existential quantifier** is a statement of the form

$(\exists x \in S)Q(x)$

(pronounced as 'there exists some x belonging to S such that $Q(x)$'), where S is a given set and $Q(x)$ is a given predicate.

The purist will write '$(\exists x \in S)Q(x)$' as

'$(\exists x)((x \in S) \wedge Q(x))$'

(pronounced as 'there exists some x such that x belongs to S and $Q(x)$'). Note that '$x \in S$', $Q(x)$ are predicates, and it makes sense to connect them to form the predicate '$(x \in S) \wedge Q(x)$'.

4. **Truth-hood of a statement of the form $(\forall x)P(x)$, and verification of such a statement.**

   The statement '$(\forall x)P(x)$' is true exactly when no matter which object x is specified, the statement $P(x)$ is a true statement.

   Recall that the statement '$(\forall x \in S)Q(x)$' is the same as '$(\forall x)((x \in S) \to Q(x))$'. Now we (have to) accept that the statement '$(\forall x \in S)Q(x)$' is true exactly when, no matter which object x is specified, if x is an element of S then the statement $Q(x)$ is a true statement.

   To verify the statement '$(\forall x \in S)Q(x)$', we proceed as described here:

   - Start with: 'Pick any object x. Suppose $x \in S$.'
     Then argue that for this fixed (but initially arbitrarily chosen) x, the statement $Q(x)$ is true.

   Examples. How do we begin the argument for the underlined conclusions in the statements below?

   (a) *Suppose* $C = \{x \mid x = n^4$ *for some* $n \in \mathbb{N}\}, D = \{x \mid x = n^2$ *for some* $n \in \mathbb{N}\}$. *Then* $\underline{C \subset D}$.
   We recall '$C \subset D$' reads:
   - 'For any object x, if $x \in C$ then $x \in D$.'

   So we proceed as:
   - ... Pick any x. Suppose $x \in C$. By the definition of C, blah-blah-blah. ... Then $x \in D$.

   (b) *Suppose* $C = \{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta)| + |\mathsf{Im}(\zeta)| < 1\}$, $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$. *Then* $\underline{C \subset D}$.
   We recall '$C \subset D$' reads:
   - 'For any object ζ, if $x \in C$ then $x \in D$.'

   So we proceed as:
   - ... Pick any ζ. Suppose $x \in C$. By the definition of C, blah-blah-blah. ... Then $x \in D$.

5. **Truth-hood of a statement of the form $(\exists x)P(x)$, and verification of such a statement.**

   The statement '$(\exists x)P(x)$' is true exactly when one object x_0 can be specified to make the statement $P(x_0)$ a true statement.

   Recall that the statement '$(\exists x \in S)Q(x)$' is the same as '$(\exists x)((x \in S) \wedge Q(x))$'. We (have to) accept that the statement '$(\exists x \in S)Q(x)$' is true exactly when, one element x_0 of S can be specified to make the statement $Q(x_0)$ is a true statement.

   To verify the statement '$(\exists x \in S)Q(x)$', we proceed as described here:

   - Name some appropriate 'candidate' x_0. Argue that $x_0 \in S$. Also argue that for this (specifically chosen) x_0, the statement $Q(x_0)$ is true.

   Such a 'candidate' x_0 may be found with whatever means. What is crucial is to verify that $x_0 \in S$ and $Q(x_0)$ is true.

   In some situations, it takes no effort to name an appropriate 'candidate'. In some situations, before naming an appropriate 'candidate', we do some roughwork to search for it. The nature of the roughwork varies from one problem to another.

   When you have no idea where to look for an appropriate 'candidate', this question may help you:

   - If $x \in S$ and $Q(x)$ is true, what can be said about x?

   Examples. How do we argue for the underlined conclusions of the statements below?

   (a) *Let* $u \in \mathbb{Z}$. *$\underline{u \text{ is divisible by } u}$.*
   Recall 'u is divisible by u' reads:
   - 'There exists some k such that $k \in \mathbb{Z}$ and $u = ku$.'

   A 'candidate' for k which we can spot immediately is 1.
   Hence we proceed as:

   ... $1 \in \mathbb{Z}$. Also, $u = 1 \cdot u$. Hence u is divisible by u.

(b) *Let $u, v, w \in \mathbb{Z}$. Suppose u is divisible by v and v is divisible by w. Then* <u>*u is divisible by w.*</u>

Recall 'u is divisible by u' reads:

- 'There exists some k such that $k \in \mathbb{Z}$ and $u = kw$.'

We want to name an appropriate 'candidate' k which, we hope, is an element of $\mathbb{Z}$ and satisfies $u = kw$.
Roughwork. We ask:

- What can be said about k if $k \in \mathbb{Z}$ and $u = kw$?

Then we 'unwrap' the assumption 'u is divisible by v and v is divisible by w' to obtain:

- 'there exist some $g, h \in \mathbb{Z}$ such that $u = gv$ and $v = hw$'.

So we obtain a 'candidate' for k, namely, $k = gh$.

In the formal argument, we proceed as:

- ... There exist some $g, h \in \mathbb{Z}$ such that $u = gv$ and $v = hw$. Take $k = gh$. Since blah-blah-blah, $k \in \mathbb{Z}$. Also, since bleh-bleh-bleh, $u = kw$. Hence u is divisible by w.

6. **Negation of a statement (starting) with one universal quantifier**.

The statement '$(\forall x)P(x)$' is true exactly when no matter which object x is specified, the statement $P(x)$ is a true statement.

Therefore the statement '$(\forall x)P(x)$' is false exactly when one x_0 can be specified to make $P(x_0)$ a false statement.

With this in mind, we accept the statements below to be logically equivalent:

(I) $\sim[(\forall x)P(x)]$.

(II) $((\forall x)P(x))$ is false.

(III) $(\exists y)(P(y)$ is false).

(IV) $(\exists y)(\sim P(y))$.

Then all the statements below are logically equivalent:

- $\sim((\forall x \in S)Q(x))$
- $\sim[(\forall x)((x \in S) \to Q(x))]$
- $(\exists x)[\sim((x \in S) \to Q(x))]$
- $(\exists x)[\sim(\sim(x \in S) \vee Q(x))]$
- $(\exists x)[(\sim(\sim(x \in S))) \wedge (\sim Q(x))]$
- $(\exists x)[(x \in S) \wedge (\sim Q(x))]$
- $(\exists x \in S)(\sim Q(x))$

In particular, $\sim((\forall x \in S)Q(x))$ and $(\exists x \in S)(\sim Q(x))$ are logically equivalent.

To dis-prove

'$(\forall x)P(x)$' (or '$(\forall x \in S)Q(x)$')

is the same as to prove

'$(\exists y)(\sim P(y))$' (or '$(\exists x \in S)(\sim Q(x))$' respectively)

This is the logical foundation of '**dis-proof by counter-example**'.

7. **Negation of a statement (starting) with one existential quantifier**.

The statement '$(\exists x)P(x)$' is true exactly when one object x_0 can be specified to make the statement $P(x_0)$ a true statement.

Therefore the statement '$(\exists x)P(x)$' is false exactly when no matter which object x is specified, the statement $P(x)$ is a false statement.

With this in mind, we also accept these statements to be logically equivalent:

(I) $\sim[(\exists x)P(x)]$.

(II) $((\exists x)P(x))$ is false.

(III) $(\forall y)(P(y)$ is false).

(IV) $(\forall y)(\sim P(y))$.

Then all the statements below are logically equivalent:

- $\sim((\exists x \in S)Q(x))$
- $\sim[(\exists x)((x \in S) \wedge Q(x))]$
- $(\forall x)[\sim((x \in S) \wedge Q(x))]$
- $(\forall x)[(\sim(x \in S)) \vee (\sim Q(x))]$
- $(\forall x)[(x \in S) \to (\sim Q(x))]$
- $(\forall x \in S)(\sim Q(x))$

In particular, $\sim((\exists x \in S)Q(x))$ and $(\forall x \in S)(\sim Q(x))$ are logically equivalent.

To dis-prove

'$(\exists x)P(x)$' (or '$(\exists x \in S)Q(x)$')

is the same as to prove

'$(\forall y)(\sim P(y))$' (or '$(\forall x \in S)(\sim Q(x))$' respectively).

In practice, we may indeed prove the negation of $(\exists x)P(x)$ (or $(\exists x \in S)Q(x)$ respectively); however, we may also proceed to obtain a contradiction from $(\exists x)P(x)$ (or $(\exists x \in S)Q(x)$ respectively).

8. **Generalized intersections and generalized unions**.

We may generalize the notions of intersection and union of sets with the help of the universal quantifier and the existential quantifier respectively.

Definition.

Let M be a set, and $\{S_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of the set M. (For any $n \in \mathbb{N}$, S_n is a subset of M.)

(1) *The* **(generalized) intersection of the infinite sequence of subsets** $\{S_n\}_{n=0}^{\infty}$ **of the set** M *is defined to be the set $\{x \in M : x \in S_n$ for any $n \in \mathbb{N}\}$. It is denoted by $\bigcap_{n=0}^{\infty} S_n$.*

(2) *The* **(generalized) union of the infinite sequence of subsets** $\{S_n\}_{n=0}^{\infty}$ **of the set** M *is defined to be the set $\{x \in M : x \in S_n$ for some $n \in \mathbb{N}\}$. It is denoted by $\bigcup_{n=0}^{\infty} S_n$.*

Remark. This is how these two definitions may be understood heuristically:

(1) Suppose $x \in M$. Then

$$\begin{aligned} x \in \bigcap_{n=0}^{\infty} S_n \quad & \text{iff} \quad (x \in S_n \text{ for any } n \in \mathbb{N}) \\ & \text{`iff'} \quad \text{`}x \in S_0 \text{ and } x \in S_1 \text{ and } x \in S_2 \text{ and ...'} \end{aligned}$$

Therefore $\bigcap_{n=0}^{\infty} S_n$ is the collection of exactly those x's belonging to M which satisfies '$x \in S_0$ and $x \in S_1$ and $x \in S_2$ and ...'.

When it happens that $S_j = M$ whenever $j \geq 2$, the set $\bigcap_{n=0}^{\infty} S_n$ is simply $S_0 \cap S_1$.

(2) Suppose $x \in M$. Then

$$\begin{aligned} x \in \bigcup_{n=0}^{\infty} S_n \quad & \text{iff} \quad (x \in S_n \text{ for some } n \in \mathbb{N}) \\ & \text{`iff'} \quad \text{`}x \in S_0 \text{ or } x \in S_1 \text{ or } x \in S_2 \text{ or ...'} \end{aligned}$$

Therefore $\bigcup_{n=0}^{\infty} S_n$ is the collection of exactly those x's belonging to M which satisfies '$x \in S_0$ or $x \in S_1$ or $x \in S_2$ or ...'.

When it happens that $S_j = \emptyset$ whenever $j \geq 2$, the set $\bigcup_{n=0}^{\infty} S_n$ is simply $S_0 \cup S_1$.

Results about intersections and unions (listed in the Handout *Set operations*) can be generalized.

Theorem ($\star$).

Let M be a set and $\{A_n\}_{n=0}^{\infty}$ be an infinite sequence of subsets of M.

(1) *Let S be a subset of M. Suppose $S \subset A_n$ for any $n \in \mathbb{N}$. Then $S \subset \bigcap_{n=0}^{\infty} A_n$.*

(2) *Let S be a subset of M. Suppose $S \subset A_n$ for some $n \in \mathbb{N}$. Then $S \subset \bigcup_{n=0}^{\infty} A_n$.*

(3) *Let T be a subset of M. Suppose $A_n \subset T$ for any $n \in \mathbb{N}$. Then $\bigcup_{n=0}^{\infty} A_n \subset T$.*

(4) *Let T be a subset of M. Suppose $A_n \subset T$ for some $n \in \mathbb{N}$. Then $\bigcap_{n=0}^{\infty} A_n \subset T$.*

(5) *Let C be a subset of M. ($\{A_n \cup C\}_{n=0}^{\infty}, \{A_n \cap C\}_{n=0}^{\infty}, \{A_n \backslash C\}_{n=0}^{\infty}, \{C \backslash A_n\}_{n=0}^{\infty}$ are infinite sequences of subsets of M.) The equalities below hold:*

(5a) $\left(\bigcap_{n=0}^{\infty} A_n\right) \cap C = \bigcap_{n=0}^{\infty} (A_n \cap C)$.

(5b) $\left(\bigcup_{n=0}^{\infty} A_n\right) \cup C = \bigcup_{n=0}^{\infty} (A_n \cup C)$.

(5c) $\left(\bigcap_{n=0}^{\infty} A_n\right) \cup C = \bigcap_{n=0}^{\infty} (A_n \cup C)$.

(5d) $\left(\bigcup_{n=0}^{\infty} A_n\right) \cap C = \bigcup_{n=0}^{\infty} (A_n \cap C)$.

(5e) $\left(\bigcap_{n=0}^{\infty} A_n\right) \backslash C = \bigcap_{n=0}^{\infty} (A_n \backslash C)$.

(5f) $\left(\bigcup_{n=0}^{\infty} A_n\right) \backslash C = \bigcup_{n=0}^{\infty} (A_n \backslash C)$.

(5g) $C \backslash \left(\bigcap_{n=0}^{\infty} A_n\right) = \bigcup_{n=0}^{\infty} (C \backslash A_n)$.

(5h) $C \backslash \left(\bigcup_{n=0}^{\infty} A_n\right) = \bigcap_{n=0}^{\infty} (C \backslash A_n)$.

The proofs of the statements in Theorem ($\star$) are left as exercises. (Statements with many quantifiers will turn up naturally in the arguments; care must be taken in their handling.)

We may further generalize the notions of intersection and union to arbitrary collections of subsets of any given set.

Definitions.

(A) *Let M be a set. The* **power set of** M *is defined to be the set $\{T \mid T$ is a subset of $M\}$. It is denoted by $\mathfrak{P}(M)$.* (So $\mathfrak{P}(M)$ is the set of all subsets of M.)

(B) *Let M be a set, and C be a subset of $\mathfrak{P}(M)$.* (So every element of C is a subset of M.)

(1) *The* **(generalized) intersection of the set** C **of subsets of the set** M *is defined to be the set $\{x \in M : x \in S$ for any $S \in C\}$. It is denoted by $\bigcap_{S \in C} S$ (with the tacit understanding $C \subset \mathfrak{P}(M)$).*

(2) *The* **(generalized) union of the set** C **of subsets of the set** M *is defined to be the set $\{x \in M : x \in S$ for some $S \in C\}$. It is denoted by $\bigcup_{S \in C} S$ (with the tacit understanding $C \subset \mathfrak{P}(M)$).*

Theorem ($\star$) can be generalized for such notions of intersection and union. Consult any standard textbook on set theory for detail.