MATH1050 Examples of proofs for properties of basic set operations.

1. Refer to the Handout *Set operations*. The statements that we are going to prove are mostly 'seemingly obvious' results in set language; for most (if not all) of them, we can convince ourselves of their validity by drawing Venn diagrams. The point here is to work out arguments which adhere to the relevant definitions, abide by the rules of logic, and need no reference to anything else.

2. Recall the respective definitions for the notions *set equality*, *subset relation*, *intersection*, *union*, *complement*.

   - *Let A, B be sets. We say that A is* **equal** *to B if both of the following statements $(\dagger), (\ddagger)$ hold:*

     $(\dagger)$ *For any object x, [if $(x \in A)$ then $(x \in B)$].*

     $(\ddagger)$ *For any object y, [if $(y \in B)$ then $(y \in A)$].*

     We write $A = B$.

   - *Let A, B be sets. We say A is a* **subset** *of B if the following statement $(\dagger)$ holds:*

     $(\dagger)$ *For any object x, [if $(x \in A)$ then $(x \in B)$].*

     We write $A \subset B$ (or $B \supset A$).

   - *Let A, B be sets.*

     $*$ *The* **intersection** *of the sets A, B is defined to be the set $\{x \mid x \in A \text{ and } x \in B\}$. It is denoted by $A \cap B$.*

     $*$ *The* **union** *of the sets A, B is defined to be the set $\{x \mid x \in A \text{ or } x \in B\}$. It is denoted by $A \cup B$.*

     $*$ *The* **complement** *of the set B in the set A is defined to be the set $\{x \mid x \in A \text{ and } x \notin B\}$. It is denoted by $A \backslash B$.*

3. **Theorem (I).**

   (3) *Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$. Then $A \subset C$.*

   Proof of Statement (3) of Theorem (I).

   Let A, B, C be sets.
   Suppose $A \subset B$ and $B \subset C$. [We want to deduce that $A \subset C$.]

   - Let x be an object. [We want to verify that if $x \in A$ then $x \in C$.]
     Suppose $x \in A$.
     Since $x \in A$ and $A \subset B$, we have $x \in B$ [according to the definition of subsets].
     Since $x \in B$ and $B \subset C$, we have $x \in C$ [according to the definition of subsets].

   It follows that $A \subset C$.

   Very formal proof of Statement (3) of Theorem (I).

   Let A, B, C be sets.

   I. Suppose $A \subset B$ and $B \subset C$. [Assumption.]
   II. $A \subset B$. [**I**.]
   III. $B \subset C$. [**I**.]
   IV. Let x be an object.
   - **IVi**. Suppose $x \in A$. [Assumption.]
   - **IVii**. $x \in B$. [**II**, **IVi**, definition of subsets.]
   - **IViii**. $x \in C$. [**III**, **IVii**, definition of subsets.]

   [We have verified that if $x \in A$ then $x \in C$.]
   V. $A \subset C$. [**IV**, definition of subsets.]

4. **Theorem (III).**

(1) *Let A be a set. $\emptyset \subset A$.*

Proof of Statement (1) of Theorem (III).

Let A be a set.

- Let x be an object.
  '$x \in \emptyset$' is a false statement.
  Therefore, [according to logic,] it is true that if $x \in \emptyset$ then $x \in A$.

It follows that $\emptyset \subset A$.

Remark. Remember that whenever H is a false statement, the conditional '$H \longrightarrow K$' will be a true statement no matter what the statement K is.

5. **Theorem (IV).**

(1) *Let A, B, S be sets. Suppose $S \subset A$ and $S \subset B$. Then $S \subset A \cap B$.*

Proof of Statement (1) of Theorem (IV).

Let A, B, S be sets.
Suppose $S \subset A$ and $S \subset B$. [We want to deduce that $S \subset A \cap B$.]

- Let x be an object. [We want to verify that if $x \in S$ then $x \in A \cap B$.]
  Suppose $x \in S$.
  Since $x \in S$ and $S \subset A$, we have $x \in A$ [according to the definition of subsets].
  Since $x \in S$ and $S \subset B$, we have $x \in B$ [according to the definition of subsets].
  Now we have $x \in A$ and $x \in B$.
  Then $x \in A \cap B$ [according to the definition of intersection].

It follows that $S \subset A \cap B$.

Very formal proof of Statement (1) of Theorem (IV).

Let A, B, S be sets.

I. Suppose $S \subset A$ and $S \subset B$. [Assumption.]
II. $S \subset A$. [**I**.]
III. $S \subset B$. [**I**.]
IV. Let x be an object.
 IVi. Suppose $x \in S$. [Assumption.]
 IVii. $x \in A$. [**II**, **IVi**, definition of subsets.]
 IViii. $x \in B$. [**III**, **IVi**, definition of subsets.]
 IViv. $x \in A$ and $x \in B$. [**IVii**, **IViii**.]
 IVv. $x \in A \cap B$. [**IVv**, definition of intersection.]
[We have verified that if $x \in S$ then $x \in A \cap B$.]
V. $S \subset A \cap B$. [**IV**, definition of subsets.]

6. **Theorem (IV).**

(2) *Let A, B, S be sets. Suppose $S \subset A$ or $S \subset B$. Then $S \subset A \cup B$.*

Proof of Statement (2) of Theorem (IV).

Let A, B, S be sets.
Suppose $S \subset A$ or $S \subset B$. [We want to deduce that $S \subset A \cup B$.]

- Let x be an object. [We want to verify that if $x \in S$ then $x \in A \cup B$.]
  Suppose $x \in S$.
  - (Case 1). Suppose $S \subset A$.
    Then, since $x \in S$ and $S \subset A$, we have $x \in A$ [according to the definition of subsets].
    Therefore $x \in A$ or $x \in B$ [by logic].
  - (Case 2). Suppose $S \subset B$.
    Then, since $x \in S$ and $S \subset B$, we have $x \in B$ [according to the definition of subsets].
    Therefore $x \in A$ or $x \in B$ [by logic].

  Therefore, in any case, $x \in A$ or $x \in B$.
  Hence $x \in A \cup B$ [according to the definition of union].

It follows that $S \subset A \cap B$.

Very formal proof of Statement (2) of Theorem (IV).

Let A, B, S be sets.

I. Suppose $S \subset A$ or $S \subset B$. [Assumption.]
II. Let x be an object.
IIi. Suppose $x \in S$. [Assumption.]
IIii.
IIii1. Suppose $S \subset A$. [One of the possibilities in **I**.]
IIii2. $x \in S$ and $S \in A$. [**IIi**, **IIii1**.]
IIii3. $x \in A$. [**IIii2**, definition of subsets.]
IIii4. $x \in A$ or $x \in B$. [**IIii3**, rules of logic.]
IIiii.
IIiii1. Suppose $S \subset B$. [The other possibility in **I**.]
IIiii2. $x \in S$ and $S \in B$. [**IIi**, **IIiii1**.]
IIiii3. $x \in B$. [**IIiii2**, definition of subsets.]
IIiii4. $x \in A$ or $x \in B$. [**IIiii3**, rules of logic.]
IIiv. $x \in A$ or $x \in B$. [**IIii**, **IIiii**.]
IIv. $x \in A \cup B$. [**IIiv**, definition of union.]
[We have verified that if $x \in S$ then $x \in A \cup B$.]
III. $S \subset A \cup B$. [**II**, definition of subsets.]

7. **Theorem (V)**.

Let A, B, C be sets. The following statements hold:

(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. (4') $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Proof of Statement (4) of Theorem (V).

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.] Suppose $x \in (A \cap B) \cup C$.
  Then $x \in A \cap B$ or $x \in C$ [according to the definition of union].
  Therefore ($x \in A$ and $x \in B$) or $x \in C$ [according to the definition of intersection].
  We now have ($x \in A$ or $x \in C$) and ($x \in B$ or $x \in C$) [by logic].
  Then ($x \in A \cup C$) and ($x \in B \cup C$) [according to the definition of union].
  Therefore $x \in (A \cup C) \cap (B \cup C)$ [according to the definition of intersection].
  [We have verified that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.] [—— ($\sharp$)]
- Let x be an object. [We want to verify that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.] Suppose $x \in (A \cup C) \cap (B \cup C)$.
  Then ($x \in A \cup C$) and ($x \in B \cup C$) [according to the definition of intersection].
  Therefore ($x \in A$ or $x \in C$) and ($x \in B$ or $x \in C$) [according to the definition of union].
  We now have ($x \in A$ and $x \in B$) or $x \in C$ [by logic].
  Then $x \in A \cap B$ or $x \in C$ [according to the definition of intersection].
  Therefore $x \in (A \cap B) \cup C$ [according to the definition of union].
  [We have verified that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.] [—— ($\flat$)]
- Hence $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ [according to ($\sharp$) and ($\flat$)].

Remark. We have used the Distributive Laws in logic. (Where?)

Very formal proof of Statement (4) of Theorem (V).

Let A, B, C be sets.

I. Let x be an object.

Ii. Suppose $x \in (A \cap B) \cup C$. [Assumption.]

Iii. $x \in A \cap B$ or $x \in C$. [**Ii**, definition of union.]

Iiii. ($x \in A$ and $x \in B$) or $x \in C$. [**Iii**, definition of intersection.]

Iiv. ($x \in A$ or $x \in C$) and ($x \in B$ or $x \in C$). [**Iiii**, rules of logic.]

Iv. ($x \in A \cup C$) and ($x \in B \cup C$). [**Iiv**, definition of union.]

Ivi. $x \in (A \cup C) \cap (B \cup C)$. [**Iv**, definition of intersection.]

[We have verified that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.]

II. Let x be an object.

IIi. Suppose $x \in (A \cup C) \cap (B \cup C)$. [Assumption.]

IIii. ($x \in A \cup C$) and ($x \in B \cup C$). [**IIi**, definition of intersection.]

IIiii. ($x \in A$ or $x \in C$) and ($x \in B$ or $x \in C$). [**IIii**, definition of union.]

IIiv. ($x \in A$ and $x \in B$) or $x \in C$. [**IIiii**, rules of logic.]

IIv. $x \in A \cap B$ or $x \in C$. [**IIiv**, definition of intersection.]

IIvi. $x \in (A \cap B) \cup C$. [**IIv**, definition of union.]

[We have verified that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.]

III. $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. [**I**, **II**, definition of equality of sets.]

Proof of Statement (4) of Theorem (V), with use of '$\Longrightarrow$'.

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.]

$$\begin{aligned}
& x \in (A \cap B) \cup C \\
\Longrightarrow\quad & x \in A \cap B \text{ or } x \in C \quad \text{[according to the definition of union]} \\
\Longrightarrow\quad & (x \in A \text{ and } x \in B) \text{ or } x \in C \quad \text{[according to the definition of intersection]} \\
\Longrightarrow\quad & (x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \quad \text{[by logic]} \\
\Longrightarrow\quad & (x \in A \cup C) \text{ and } (x \in B \cup C) \quad \text{[according to the definition of union]} \\
\Longrightarrow\quad & x \in (A \cup C) \cap (B \cup C) \quad \text{[according to the definition of intersection]}
\end{aligned}$$

  [We have verified that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.] [—— ($\sharp$)]

- Let x be an object. [We want to verify that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.]

$$\begin{aligned}
& x \in (A \cup C) \cap (B \cup C) \\
\Longrightarrow\quad & (x \in A \cup C) \text{ and } (x \in B \cup C) \quad \text{[according to the definition of intersection]} \\
\Longrightarrow\quad & (x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \quad \text{[according to the definition of union]} \\
\Longrightarrow\quad & (x \in A \text{ and } x \in B) \text{ or } x \in C \quad \text{[by logic]} \\
\Longrightarrow\quad & x \in A \cap B \text{ or } x \in C \quad \text{[according to the definition of intersection]} \\
\Longrightarrow\quad & x \in (A \cap B) \cup C \quad \text{[according to the definition of union]}
\end{aligned}$$

  [We have verified that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.] [—— ($\flat$)]

- Hence $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ [according to ($\sharp$) and ($\flat$)].

Remark. Whenever you write $H \Longrightarrow K$, you are telling the reader that, the statement H is (known/asserted to be) true and the conditional '$H \longrightarrow K$' is also (known to be) true, and so by Modus Ponens you conclude that the statement K is true. (So whenever there is the temptation to write 'blah-blah-blah $\Longrightarrow$ bleh-bleh-bleh', think whether you mean the above.)

Proof of Statement (4) of Theorem (V), with use of '$\Longleftrightarrow$'.

Let A, B, C be sets.

- Let x be an object.
  [We want to verify that $x \in (A \cap B) \cup C$ iff $x \in (A \cup C) \cap (B \cup C)$.]

$$\begin{aligned}
& x \in (A \cap B) \cup C \\
\Longleftrightarrow\quad & x \in A \cap B \text{ or } x \in C \\
\Longleftrightarrow\quad & (x \in A \text{ and } x \in B) \text{ or } x \in C \\
\Longleftrightarrow\quad & (x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \\
\Longleftrightarrow\quad & (x \in A \cup C) \text{ and } (x \in B \cup C) \\
\Longleftrightarrow\quad & x \in (A \cup C) \cap (B \cup C)
\end{aligned}$$

  [We have verified that $x \in (A \cap B) \cup C$ iff $x \in (A \cup C) \cap (B \cup C)$.]

It follows that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

8. **Theorem (V).**

Let A, B, C be sets. The following statements hold:

(6) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.   (6') $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

Proof of Statement (6) of Theorem (V).

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in A \backslash (B \cap C)$ then $x \in (A \backslash B) \cup (A \backslash C)$.] Suppose $x \in A \backslash (B \cap C)$.
  Then $x \in A$ and $x \notin B \cap C$ [according to the definition of complement].
  Therefore $x \in A$ and (it is not true that $x \in B \cap C$).
  Hence $x \in A$ and (it is not true that ($x \in B$ and $x \in C$)) [according to the definition of intersection].
  Then $x \in A$ and ((it is not true that $x \in B$) or (it is not true that $x \in C$)) [by logic].
  We now have $x \in A$ and ($x \notin B$ or $x \notin C$).
  Then ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$) [by logic].
  Therefore $x \in A \backslash B$ or $x \in A \backslash C$ [according to the definition of complement].
  Hence $x \in (A \backslash B) \cup (A \backslash C)$ [according to the definition of union].
  [We have verified that if $x \in A \backslash (B \cap C)$ then $x \in (A \backslash B) \cup (A \backslash C)$.] [—— ($\sharp$)]
- Let x be an object. [We want to verify that if $x \in (A \backslash B) \cup (A \backslash C)$ then $x \in A \backslash (B \cap C)$.] Suppose $x \in (A \backslash B) \cup (A \backslash C)$.
  Then $x \in A \backslash B$ or $x \in A \backslash C$ [according to the definition of union].
  Therefore ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$) [according to the definition of complement].
  We now have $x \in A$ and ($x \notin B$ or $x \notin C$) [by logic].
  Then $x \in A$ and ((it is not true that $x \in B$) or (it is not true that $x \in C$)).
  Therefore $x \in A$ and (it is not true that ($x \in B$ and $x \in C$)) [by logic].
  Hence $x \in A$ and (it is not true that $x \in B \cap C$) [according to the definition of intersection].
  We have $x \in A$ and $x \notin B \cap C$.
  Then $x \in A \backslash (B \cap C)$ [according to the definition of complement].
  [We have verified that if $x \in (A \backslash B) \cup (A \backslash C)$ then $x \in A \backslash (B \cap C)$.] [—— ($\flat$)]
- Hence $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$ [according to ($\sharp$) and ($\flat$)].

Remark. We have used De Morgan's Laws in logic. (Where?)

Proof of Statement (6) of Theorem (V), with use of '$\Longleftrightarrow$'.

Let A, B, C be sets.

- Let x be an object.
  [We want to verify that $x \in A \backslash (B \cap C)$ iff $x \in (A \backslash B) \cup (A \backslash C)$.]

$$
\begin{array}{ll}
 & x \in A \backslash (B \cap C) \\
\Longleftrightarrow & x \in A \text{ and } x \notin B \cap C \\
\Longleftrightarrow & x \in A \text{ and (it is not true that } x \in B \cap C) \\
\Longleftrightarrow & x \in A \text{ and (it is not true that } (x \in B \text{ and } x \in C)) \\
\Longleftrightarrow & x \in A \text{ and ((it is not true that } x \in B) \text{ or (it is not true that } x \in C)) \\
\Longleftrightarrow & x \in A \text{ and } (x \notin B \text{ or } x \notin C) \\
\Longleftrightarrow & (x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C) \\
\Longleftrightarrow & x \in A \backslash B \text{ or } x \in A \backslash C \\
\Longleftrightarrow & x \in (A \backslash B) \cup (A \backslash C)
\end{array}
$$

  [We have verified that $x \in A \backslash (B \cap C)$ iff $x \in (A \backslash B) \cup (A \backslash C)$.]

It follows that $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.