1. Refer to the Handout *Set operations*.

2. Recall the respective definitions for the notions *set equality*, *subset relation*, *intersection*, *union*, *complement*.

- *Let A, B be sets. We say A is* **equal** *to B if both of the following statements* $(\dagger), (\ddagger)$ *hold:*

  $(\dagger)$ *For any object x, [if $(x \in A)$ then $(x \in B)$].*

  $(\ddagger)$ *For any object y, [if $(y \in B)$ then $(y \in A)$].*

  We write $A = B$.

- *Let A, B be sets. We say A is a* **subset** *of B if the following statement* $(\dagger)$ *holds:*

  $(\dagger)$ *For any object x, [if $(x \in A)$ then $(x \in B)$].*

  We write $A \subset B$ (or $B \supset A$).

- *Let A, B be sets.*
  - $*$ *The* **intersection** *of the sets A, B is defined to be the set* $\{x \mid x \in A \text{ and } x \in B\}$. *It is denoted by $A \cap B$.*
  - $*$ *The* **union** *of the sets A, B is defined to be the set* $\{x \mid x \in A \text{ or } x \in B\}$. *It is denoted by $A \cup B$.*
  - $*$ *The* **complement** *of the set B in the set A is defined to be the set* $\{x \mid x \in A \text{ and } x \notin B\}$. *It is denoted by $A \backslash B$.*

3. **Theorem (I).**

(3) *Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$. Then $A \subset C$.*

Proof of Statement (3) of Theorem (I).

Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$.

'Unwrapped': 'For any y, if $y \in A$ then $y \in B$.'

'Unwrapped': 'For any z, if $z \in B$ then $z \in C$.'

[Want to deduce: $A \subset C$.
What? 'For any x, if $x \in A$ then $x \in C$.']

- Let x be an object.
  Suppose $x \in A$.
  Since $x \in A$ and $A \subset B$, we have $x \in B$ [according to the definition of subsets].
  Since $x \in B$ and $B \subset C$, we have $x \in C$ [according to the definition of subsets].

It follows that $A \subset C$. □

Very formal proof of Statement (3) of Theorem (I).

Let A, B, C be sets.

I. Suppose $A \subset B$ and $B \subset C$. [Assumption.]

II. $A \subset B$. [**I**.]

III. $B \subset C$. [**I**.]

IV. Let x be an object.

IVi. Suppose $x \in A$. [Assumption.]

IVii. $x \in B$. [**II**, **IVi**, definition of subsets.]

IViii. $x \in C$. [**III**, **IVii**, definition of subsets.]

[We have verified that if $x \in A$ then $x \in C$.]

V. $A \subset C$. [**IV**, definition of subsets.]

4. **Theorem (III).**

(1) *Let A be a set. $\emptyset \subset A$.*

Proof of Statement (1) of Theorem (III).

Let A be a set.

[Want to deduce: $\phi \subset A$.
What? 'For any x, if $x \in \phi$ then $x \in A$'.]

???

- Let x be an object.
  '$x \in \phi$' is a false statement.
  Therefore, [according to logic], it is true that
  if $x \in \phi$ then $x \in A$.

It follows that $\phi \subset A$. □

Remark.

Remember that whenever H is a false statement, the conditional

$$\text{`}H \longrightarrow K\text{'}$$

will be a true statement no matter what the statement K is.

H	K	$H \to K$
T	T	T
T	F	F
F	T	T
F	F	T

5. **Theorem (IV).**

(1) *Let A, B, S be sets. Suppose $S \subset A$ and $S \subset B$. Then $S \subset A \cap B$.*

Proof of Statement (1) of Theorem (IV).

'For any y, if $y \in S$ then $y \in A$.'

'For any z, if $z \in S$ then $z \in B$.'

Let A, B, S be sets.

Suppose $S \subset A$ and $S \subset B$. [We want to deduce that $S \subset A \cap B$.]

'For any x, if $x \in S$ then $x \in A \cap B$.'

- Let x be an object.

  Suppose $x \in S$. [Want to deduce: $x \in A \cap B$.
  What is it? '$x \in A$ and $x \in B$.']

  Since $x \in S$ and $S \subset A$, we have $x \in A$ [according to the definition of subsets].

  Since $x \in S$ and $S \subset B$, we have $x \in B$ [according to the definition of subsets].

  Now we have $x \in A$ and $x \in B$ (simultaneously).

  Then $x \in A \cap B$ [according to the definition of intersection].

  It follows that $S \subset A \cap B$. □

Theorem (IV).

(1) *Let A, B, S be sets. Suppose $S \subset A$ and $S \subset B$. Then $S \subset A \cap B$.*

Proof of Statement (1) of Theorem (IV). ...

Very formal proof of Statement (1) of Theorem (IV).

Let A, B, S be sets.

I. Suppose $S \subset A$ and $S \subset B$. [Assumption.]

II. $S \subset A$. [**I**.]

III. $S \subset B$. [**I**.]

IV. Let x be an object.

IVi. Suppose $x \in S$. [Assumption.]

IVii. $x \in A$. [**II**, **IVi**, definition of subsets.]

IViii. $x \in B$. [**III**, **IVi**, definition of subsets.]

IViv. $x \in A$ and $x \in B$. [**IVii**, **IViii**.]

IVv. $x \in A \cap B$. [**IVv**, definition of intersection.]

[We have verified that if $x \in S$ then $x \in A \cap B$.]

V. $S \subset A \cap B$. [**IV**, definition of subsets.]

6. **Theorem (IV).**

(2) *Let A, B, S be sets. Suppose $S \subset A$ or $S \subset B$. Then $S \subset A \cup B$.*

Proof of Statement (2) of Theorem (IV).

'For any y, if $y \in S$ then $y \in A$.'

'For any z, if $z \in S$ then $z \in B$.'

Let A, B, S be sets.

Suppose $S \subset A$ or $S \subset B$. [We want to deduce that $S \subset A \cup B$.]

'For any x, if $x \in S$ then $x \in A \cup B$.'

- Let x be an object.

  Suppose $x \in S$. [Want to deduce: $x \in A \cup B$. What is it? '$x \in A$ or $x \in B$'.]

  * (Case 1). Suppose $S \subset A$.

    Then, since $x \in S$ and $S \subset A$, we have $x \in A$ [according to the definition of subsets].

    Therefore $x \in A$ or $x \in B$.

  * (Case 2). Suppose $S \subset B$

    Then, since $x \in S$ and $S \subset B$, we have $x \in B$ [according to the definition of subsets].

    Therefore $x \in A$ or $x \in B$.

  Therefore, in any case, $x \in A$ or $x \in B$.

  Hence $x \in A \cup B$ [according to the definition of union].

It follows that $S \subset A \cup B$. □

Theorem (IV).

(2) *Let A, B, S be sets. Suppose $S \subset A$ or $S \subset B$. Then $S \subset A \cup B$.*

Proof of Statement (2) of Theorem (IV). ...

Very formal proof of Statement (2) of Theorem (IV).

Let A, B, S be sets.

I. Suppose $S \subset A$ or $S \subset B$. [Assumption.]

II. Let x be an object.

IIi. Suppose $x \in S$. [Assumption.]

IIii.

IIii1. Suppose $S \subset A$. [One of the possibilities in **I**.]

IIii2. $x \in S$ and $S \in A$. [**IIi**, **IIii1**.]

IIii3. $x \in A$. [**IIii2**, definition of subsets.]

IIii4. $x \in A$ or $x \in B$. [**IIii3**, rules of logic.]

IIiii.

IIiii1. Suppose $S \subset B$. [The other possibility in **I**.]

IIiii2. $x \in S$ and $S \in B$. [**IIi**, **IIiii1**.]

IIiii3. $x \in B$. [**IIiii2**, definition of subsets.]

IIiii4. $x \in A$ or $x \in B$. [**IIiii3**, rules of logic.]

IIiv. $x \in A$ or $x \in B$. [**IIii**, **IIiii**.]

IIv. $x \in A \cup B$. [**IIiv**, definition of union.]

[We have verified that if $x \in S$ then $x \in A \cup B$.]

III. $S \subset A \cup B$. [**II**, definition of subsets.]

7. **Theorem (V).**

Let A, B, C be sets. The following statements hold:

(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. (4') $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Proof of Statement (4) of Theorem (V).

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.]
  Suppose $x \in (A \cap B) \cup C$.
  Then $x \in A \cap B$ or $x \in C$ [according to the definition of union].
  Therefore ($x \in A$ and $x \in B$) or $x \in C$ [according to the definition of intersection].
  (Hence) We now have ($x \in A$ or $x \in C$) and ($x \in B$ or $x \in C$) [by logic]. ← Distributive Laws for Conjunction and Disjunction
  Then ($x \in A \cup C$) and ($x \in B \cup C$) [according to the definition of union].
  Therefore $x \in (A \cup C) \cap (B \cup C)$ [according to the definition of intersection].
  [We have verified that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.] [—— ($\sharp$)]
- Let x be an object. [We want to verify that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.] ...
- Hence $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ [according to ($\sharp$) and ($\flat$)].

Remark. We have used the Distributive Laws in logic. (Where?)

Theorem (V).

Let A, B, C be sets. The following statements hold:

(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. (4') $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Proof of Statement (4) of Theorem (V).

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.] ...
- Let x be an object. [We want to verify that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.]
  Suppose $x \in (A \cup C) \cap (B \cup C)$.
  Then $(x \in A \cup C)$ and $(x \in B \cup C)$ [according to the definition of intersection].
  Therefore $(x \in A$ or $x \in C)$ and $(x \in B$ or $x \in C)$ [according to the definition of union].
  (Hence) We now have $(x \in A$ and $x \in B)$ or $x \in C$ [by logic]. ← Distributive Laws for Conjunction and Disjunction
  Then $x \in A \cap B$ or $x \in C$ [according to the definition of intersection].
  Therefore $x \in (A \cap B) \cup C$ [according to the definition of union].
  [We have verified that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.] [—— ($\flat$)]
- Hence $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ [according to ($\sharp$) and ($\flat$)].

Remark. We have used the Distributive Laws in logic. (Where?)

Very formal proof of Statement (4) of Theorem (V)?

Compare the orders of the lines in the respective parts of the proof.

Theorem (V).

Let A, B, C be sets. The following statements hold:

(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. (4') $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Proof of Statement (4) of Theorem (V), with use of '$\Longrightarrow$'.

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.]

$$
\begin{array}{ll}
 & x \in (A \cap B) \cup C \\
\Longrightarrow & x \in A \cap B \text{ or } x \in C \quad \text{[according to the definition of union]} \\
\Longrightarrow & (x \in A \text{ and } x \in B) \text{ or } x \in C \quad \text{[according to the definition of intersection]} \\
\Longrightarrow & (x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \quad \text{[by logic]} \\
\Longrightarrow & (x \in A \cup C) \text{ and } (x \in B \cup C) \quad \text{[according to the definition of union]} \\
\Longrightarrow & x \in (A \cup C) \cap (B \cup C) \quad \text{[according to the definition of intersection]}
\end{array}
$$

  [We have verified that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.] [—— ($\sharp$)]

- Let x be an object. [We want to verify that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.]

  ...

- Hence $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ [according to ($\sharp$) and ($\flat$)].

Remark on the use of '$\Longrightarrow$'.

Whenever you write $H \Longrightarrow K$, you are telling the reader that,

* the statement H is (known/asserted to be) true and
* the conditional '$H \longrightarrow K$' is also (known to be) true,
* and so by **Modus Ponens** you conclude that the statement K is true.

(So whenever there is the temptation to write

$$\text{`blah-blah-blah} \Longrightarrow \text{bleh-bleh-bleh'},$$

think whether you mean the above.)

Theorem (V).

Let A, B, C be sets. The following statements hold:

(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. (4') $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Proof of Statement (4) of Theorem (V), with use of '$\Longrightarrow$'.

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in (A \cap B) \cup C$ then $x \in (A \cup C) \cap (B \cup C)$.]

  ...

- Let x be an object. [We want to verify that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.]

$$
\begin{aligned}
& x \in (A \cup C) \cap (B \cup C) \\
\Longrightarrow\ & (x \in A \cup C) \text{ and } (x \in B \cup C) \quad \text{[according to the definition of intersection]} \\
\Longrightarrow\ & (x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \quad \text{[according to the definition of union]} \\
\Longrightarrow\ & (x \in A \text{ and } x \in B) \text{ or } x \in C \quad \text{[by logic]} \\
\Longrightarrow\ & x \in A \cap B \text{ or } x \in C \quad \text{[according to the definition of intersection]} \\
\Longrightarrow\ & x \in (A \cap B) \cup C \quad \text{[according to the definition of union]}
\end{aligned}
$$

  [We have verified that if $x \in (A \cup C) \cap (B \cup C)$ then $x \in (A \cap B) \cup C$.] [—— ($\flat$)]

- Hence $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$ [according to ($\sharp$) and ($\flat$)].

Theorem (V).
Let A, B, C be sets. The following statements hold:

(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$. (4') $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

Proof of Statement (4) of Theorem (V), with use of '$\Longleftrightarrow$'.
Let A, B, C be sets.

- Let x be an object.
  [We want to verify that $x \in (A \cap B) \cup C$ iff $x \in (A \cup C) \cap (B \cup C)$.]

$$
\begin{aligned}
& x \in (A \cap B) \cup C \\
\Longleftrightarrow \quad & x \in A \cap B \text{ or } x \in C \\
\Longleftrightarrow \quad & (x \in A \text{ and } x \in B) \text{ or } x \in C \\
\Longleftrightarrow \quad & (x \in A \text{ or } x \in C) \text{ and } (x \in B \text{ or } x \in C) \\
\Longleftrightarrow \quad & (x \in A \cup C) \text{ and } (x \in B \cup C) \\
\Longleftrightarrow \quad & x \in (A \cup C) \cap (B \cup C)
\end{aligned}
$$

  [We have verified that $x \in (A \cap B) \cup C$ iff $x \in (A \cup C) \cap (B \cup C)$.]

It follows that $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

8. **Theorem (V).**

Let A, B, C be sets. The following statements hold:

(6) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$. (6') $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

Proof of Statement (6) of Theorem (V).

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in A \backslash (B \cap C)$ then $x \in (A \backslash B) \cup (A \backslash C)$.]
  Suppose $x \in A \backslash (B \cap C)$.
  Then $x \in A$ and $x \notin B \cap C$ [according to the definition of complement].
  Therefore $x \in A$ and (it is not true that $x \in B \cap C$).
  Hence $x \in A$ and (it is not true that ($x \in B$ and $x \in C$)) [according to the definition of intersection].
  Then $x \in A$ and ((it is not true that $x \in B$) or (it is not true that $x \in C$)) [by logic]. ← De Morgan's Laws
  We now have $x \in A$ and ($x \notin B$ or $x \notin C$).
  Then ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$) [by logic]. ← Distributive Laws for Conjunction and Disjunction
  Therefore $x \in A \backslash B$ or $x \in A \backslash C$ [according to the definition of complement].
  Hence $x \in (A \backslash B) \cup (A \backslash C)$ [according to the definition of union].
  [We have verified that if $x \in A \backslash (B \cap C)$ then $x \in (A \backslash B) \cup (A \backslash C)$.] [—— ($\sharp$)]
- Let x be an object. [We want to verify that if $x \in (A \backslash B) \cup (A \backslash C)$ then $x \in A \backslash (B \cap C)$.] ...
- Hence $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$ [according to ($\sharp$) and ($\flat$)].

Remark. We have used De Morgan's Laws in logic. (Where?)

Theorem (V).

Let A, B, C be sets. The following statements hold:

(6) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$. (6') $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

Proof of Statement (6) of Theorem (V).

Let A, B, C be sets.

- Let x be an object. [We want to verify that if $x \in A \backslash (B \cap C)$ then $x \in (A \backslash B) \cup (A \backslash C)$.] ...
- Let x be an object. [We want to verify that if $x \in (A \backslash B) \cup (A \backslash C)$ then $x \in A \backslash (B \cap C)$.]

  Suppose $x \in (A \backslash B) \cup (A \backslash C)$.

  Then $x \in A \backslash B$ or $x \in A \backslash C$ [according to the definition of union].

  Therefore ($x \in A$ and $x \notin B$) or ($x \in A$ and $x \notin C$) [according to the definition of complement].

  (Hence) We now have $x \in A$ and ($x \notin B$ or $x \notin C$) [by logic]. ← Distributive Laws for Conjunction and Disjunction

  Then $x \in A$ and ((it is not true that $x \in B$) or (it is not true that $x \in C$)).

  Therefore $x \in A$ and (it is not true that ($x \in B$ and $x \in C$)) [by logic]. ← De Morgan's Laws

  Hence $x \in A$ and (it is not true that $x \in B \cap C$) [according to the definition of intersection].

  We have $x \in A$ and $x \notin B \cap C$.

  Then $x \in A \backslash (B \cap C)$ [according to the definition of complement].

  [We have verified that if $x \in (A \backslash B) \cup (A \backslash C)$ then $x \in A \backslash (B \cap C)$.] [—— ($\flat$)]
- Hence $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$ [according to ($\sharp$) and ($\flat$)].

Remark. We have used De Morgan's Laws in logic. (Where?)

Theorem (V).

Let A, B, C be sets. The following statements hold:

(6) $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$. (6') $A\backslash(B \cup C) = (A\backslash B) \cap (A\backslash C)$.

Proof of Statement (6) of Theorem (V), with use of '$\Longleftrightarrow$'.

Let A, B, C be sets.

- Let x be an object. [We want to verify that $x \in A\backslash(B \cap C)$ iff $x \in (A\backslash B) \cup (A\backslash C)$.]

$$
\begin{aligned}
& x \in A\backslash(B \cap C) \\
\Longleftrightarrow\ & x \in A \text{ and } x \notin B \cap C \\
\Longleftrightarrow\ & x \in A \text{ and (it is not true that } x \in B \cap C) \\
\Longleftrightarrow\ & x \in A \text{ and } \underline{\text{(it is not true that } (x \in B \text{ and } x \in C))} \\
\Longleftrightarrow\ & x \in A \text{ and } \underline{\text{((it is not true that } x \in B) \text{ or (it is not true that } x \in C))} \\
\Longleftrightarrow\ & \underline{x \in A \text{ and } (x \notin B \text{ or } x \notin C)} \\
\Longleftrightarrow\ & \underline{(x \in A \text{ and } x \notin B) \text{ or } (x \in A \text{ and } x \notin C)} \\
\Longleftrightarrow\ & x \in A\backslash B \text{ or } x \in A\backslash C \\
\Longleftrightarrow\ & x \in (A\backslash B) \cup (A\backslash C)
\end{aligned}
$$

[We have verified that $x \in A\backslash(B \cap C)$ iff $x \in (A\backslash B) \cup (A\backslash C)$.]

It follows that $A\backslash(B \cap C) = (A\backslash B) \cup (A\backslash C)$.