MATH1050 Examples of proofs concerned with 'subset relations'

1. Recall the **Method of specification (for the construction of sets)**:

   Suppose A is a set and $P(x)$ is a predicate with variable x.

   - $\{x \mid P(x)\}$ refers to the set (if it is indeed a set) which contains exactly every object x
     * for which the statement $P(x)$ is true.
   - $\{x \in A : P(x)\}$ refers to the set which contains exactly every object x
     * which is an element of the given set A and
     * for which the statement $P(x)$ is true.

     By definition it is a subset of A.

2. Recall the notions of *set equality* and *subset relations*:

   - *Let A, B be sets. We say A is* **equal** *to B if both of the following statements $(\dagger), (\ddagger)$ hold:*

     $(\dagger)$ *For any object x, [if $(x \in A)$ then $(x \in B)$].*

     $(\ddagger)$ *For any object y, [if $(y \in B)$ then $(y \in A)$].*

     We write $A = B$.
   - *Let A, B be sets. We say A is a* **subset** *of B if the following statement $(\dagger)$ holds:*

     $(\dagger)$ *For any object x, [if $(x \in A)$ then $(x \in B)$].*

     We write $A \subset B$ (or $B \supset A$).

3. **Question**. *What do we mean by 'A is not a subset of B'?*

   Answer. A is not a subset of B exactly when some element of A fails to be an element of B. Though more formal, a more useful formulation for the same thing is:

   - *There exists some object x_0 such that ($x_0 \in A$ and $x_0 \notin B$).*

   In this situation, we write $A \not\subset B$.

4. **Example (a)**.

   Let $C = \{x \mid x = n^4$ for some $n \in \mathbb{N}\}$, $D = \{x \mid x = n^2$ for some $n \in \mathbb{N}\}$.

   The statements below hold:

   (1) $C \subset D$. (2) $D \not\subset C$.

   Heuristic ideas for the statements:

   - C is the set of all biquadratic numbers while D is the set of all square numbers.
   - Every biquadratic number is the square of a square number. So we expect '$C \subset D$' to hold.
   - There may be some square number which is not a biquadratic number; for instance, the square of a non-square number. So we expect '$D \not\subset C$' to hold.

   It is important to have these ideas before we start writing down the proofs: indeed, they are the core ideas in the proofs. But it is equally important to organize these ideas to give a coherent argument for the respective statements, with reference to the definitions of the sets C, D, and the definition for the notion of subset relation.

5. **Proofs of the statements in Example (a)**.

   (1) [*Reminder.* We want to prove '*for any x, if $x \in C$ then $x \in D$*'.]

   Pick any object x, Suppose $x \in C$.

   [*Roughwork.* What to deduce? '$x \in D$'. What does it read? 'Unwrap' '$x \in D$' to see what it is.
   How to reach '$x \in D$'? 'Unwrap' '$x \in C$' to see what may help us.]

   There exists some $n \in \mathbb{N}$ such that $x = n^4$.
   Take $m = n^2$. Note that $m \in \mathbb{N}$.
   We have $x = m^2$ and $m \in \mathbb{N}$.
   Hence $x \in D$.
   It follows that $C \subset D$.

(2) [*Preparation.* find out what is to be done. We want to prove that there exists some x_0 such that $x_0 \in D$ and $x_0 \notin C$. (This is an existence statement.)
So we look for an appropriate x_0. Does our heuristic understanding of C, D in this specific example help us spot a candidate? Is such a candidate a 'good one'?]

Take $x_0 = 4$.
Note that $x_0 = 2^2$ and $2 \in \mathbb{N}$.
Then $x_0 \in D$.
Claim: $x_0 \notin C$.
Justification of this claim (with the help of proof-by-contradiction):

* Suppose it were true that $x_0 \in C$.
Then there would exist some $n \in \mathbb{N}$ such that $x_0 = n^4$.
Now $4 = n^4$. Since $n \in \mathbb{R}$ and $n \geq 0$, we would have $n = \sqrt{2}$.
But $\sqrt{2} \notin \mathbb{N}$. Contradiction arises.
Hence $x_0 \notin C$ in the first place.

It follows that $D \not\subset C$.

6. Below are other examples similar to Example (a).

Example (b).

Let $C = \{x \mid x = r^4 \text{ for some } r \in \mathbb{Q}\}, D = \{x \mid x = r^2 \text{ for some } r \in \mathbb{Q}\}$.

The statements below hold:

(1) $C \subset D$. (2) $D \not\subset C$.

Example (c).

Let $C = \{x \mid x = s + t\sqrt{2} \text{ for some } s, t \in \mathbb{Z}\}, D = \{x \mid x = u + v\sqrt{3} \text{ for some } u, v \in \mathbb{Z}\}$.

The statements below hold:

(1) $\mathbb{Z} \subset C \cap D$. (2) $C \not\subset D$. (3) $D \not\subset C$. (4) $C \cap D \subset \mathbb{Z}$. (5) $C \cap D = \mathbb{Z}$.

Example (d).

Let $C = \{x \mid x = s + t\sqrt{2} \text{ for some } s, t \in \mathbb{Q}\}, D = \{x \mid x = u + v\sqrt{3} \text{ for some } u, v \in \mathbb{Q}\}$.

The statements below hold:

(1) $\mathbb{Q} \subset C \cap D$. (2) $C \not\subset D$. (3) $D \not\subset C$. (4) $C \cap D \subset \mathbb{Q}$. (5) $C \cap D = \mathbb{Q}$.

7. **Example (e).**

Let $C = \{\zeta \in \mathbb{C} : |\mathsf{Re}(\zeta)| + |\mathsf{Im}(\zeta)| < 1\}$, $D = \{\zeta \in \mathbb{C} : |\zeta| < 1\}$.

The statements below hold:

(1) $C \subset D$. (2) $D \not\subset C$.

Heuristic ideas for the statements, which can be visualized using the Argand plane:

- C is the 'open' square with vertices at $1, i, -1, -i$ while D is the 'open' unit disc centred at 0.
- Every point in C is of a distance less than 1 from the point 0. So we expect '$C \subset D$' to hold.
- There may be some complex number in D lying outside C. So we expect '$D \not\subset C$' to hold.

The question is: how to organize these ideas to give a coherent argument for the respective statements, with reference to the definitions of the sets C, D, and the definition for the notion of subset relation, and with the help of the algebra for complex numbers?

8. **Proofs of the statements in Example (e).**

(1) [*Reminder.* We want to prove '*for any* $\zeta \in \mathbb{C}$, *if* $\zeta \in C$ *then* $\zeta \in D$'.]

Pick any complex number ζ. Suppose $\zeta \in C$.

[*Roughwork.* What to deduce? '$\zeta \in D$'. What does it read? '$|\zeta| < 1$.'
How to reach '$\zeta \in D$'? Find out what '$\zeta \in C$' reads: it is $|\mathsf{Re}(\zeta)| + |\mathsf{Im}(\zeta)| < 1$.]

Then $|\mathsf{Re}(\zeta)| + |\mathsf{Im}(\zeta)| < 1$.

[*Ask.* How does $|\zeta|$ link up with $|\mathsf{Re}(\zeta)|, |\mathsf{Im}(\zeta)|$?]

We have $|\zeta|^2 = (\mathsf{Re}(\zeta))^2 + (\mathsf{Im}(\zeta))^2 = |\mathsf{Re}(\zeta)|^2 + |\mathsf{Im}(\zeta)|^2 \leq 1 \cdot |\mathsf{Re}(\zeta)| + 1 \cdot |\mathsf{Im}(\zeta)| = |\mathsf{Re}(\zeta)| + |\mathsf{Im}(\zeta)| < 1$.
Then $|\zeta| < 1$. Therefore $\zeta \in D$.
It follows that $C \subset D$.

(2) [*Preparation.* find out what is to be done. We want to prove that there exists some ζ_0 such that $\zeta_0 \in D$ and $\zeta_0 \notin C$. (This is an existence statement.)
So we look for an appropriate ζ_0. Does our heuristic understanding of C, D in this specific example help us spot a candidate? Is such a candidate a 'good one'?]

Take $\zeta_0 = \dfrac{1+i}{2}$.

Note that $|\zeta_0|^2 = (\mathsf{Re}(\zeta_0))^2 + (\mathsf{Im}(\zeta_0))^2 = \dfrac{1}{4} + \dfrac{1}{4} = \dfrac{1}{2} < 1$. Then $|\zeta_0| < 1$. Therefore $\zeta_0 \in D$.

Note that $|\mathsf{Re}(\zeta_0)| + |\mathsf{Im}(\zeta_0)| = \dfrac{1}{2} + \dfrac{1}{2} = 1$. Then $\zeta_0 \notin C$.

It follows that $D \not\subset C$.

9. Below are other examples similar to Example (e).

Example (f).

Let $C = \{\zeta \in \mathbb{C} : |\zeta - 1| \leq 1\}, D = \{\zeta \in \mathbb{C} : |\zeta| \leq 2\}$.

The statements below hold:

(1) $C \subset D$. (2) $D \not\subset C$.

Example (g).

Let $C = \{\zeta \in \mathbb{C} : \mathsf{Re}(\zeta) \geq 0\}, D = \{\zeta \in \mathbb{C} : \mathsf{Im}(\zeta) \geq 0\}, E = \{\zeta \in \mathbb{C} : |\zeta - 1 - i| \leq 1\}$.

The statements below hold:

(1) $E \subset C \cap D$. (2) $C \not\subset D$. (3) $D \not\subset C$.

Example (h).

Let $C = \{\zeta \in \mathbb{C} : |\zeta - 4| < 5\}, D = \{\zeta \in \mathbb{C} : |\zeta + 4| < 5\}, E = \{\zeta \in \mathbb{C} : |\zeta| < 3\}$.

The statements below hold:

(1) $C \not\subset D$. (2) $D \not\subset C$. (3) $E \not\subset C$. (4) $E \not\subset D$. (5) $E \subset C \cup D$.

10. **Example (i).**

Let G be an $(m \times n)$-matrix with real entries, and H be an $(n \times p)$-matrix with real entries.

The statements below hold:

(1) *The null space of H is a subset of the null space of GH.*

(2) *Suppose the null space of G is $\{\mathbf{0}_n\}$. Then the null space of GH is a subset of the null space of H.*

Remark. *The null space $\mathcal{N}(K)$ of a $(p \times q)$-matrix K with real entries is defined by* $\mathcal{N}(K) = \{\mathbf{v} \in \mathbb{R}^q : K\mathbf{v} = \mathbf{0}_p\}$.

11. **Proofs of the statements in Example (i).**

(1) [*Reminder.* We want to prove '*for any* $\mathbf{x} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \mathcal{N}(H)$ *then* $\mathbf{x} \in \mathcal{N}(GH)$'.]

Pick any $\mathbf{x} \in \mathbb{R}^p$. Suppose $\mathbf{x} \in \mathcal{N}(H)$.

[*Roughwork.* What to deduce? '$\mathbf{x} \in \mathcal{N}(GH)$'. What does it read? '$(GH)\mathbf{x} = \mathbf{0}_m$.'
How to reach '$(GH)\mathbf{x} = \mathbf{0}_m$'? Find out what '$\mathbf{x} \in \mathcal{N}(H)$' reads: it is '$H\mathbf{x} = \mathbf{0}_n$'.]

Then by the definition of $\mathcal{N}(H)$, we have $H\mathbf{x} = \mathbf{0}_n$.
Therefore $(GH)\mathbf{x} = G(H\mathbf{x}) = G\mathbf{0}_n = \mathbf{0}_m$.
Hence, by the definition of $\mathcal{N}(GH)$, we have $\mathbf{x} \in \mathcal{N}(GH)$.
It follows that $\mathcal{N}(H) \subset \mathcal{N}(GH)$.

(2) Suppose the null space of G is $\{\mathbf{0}_n\}$.

[*Reminder.* We want to deduce, under the above assumption, that '$\mathcal{N}(GH) \subset \mathcal{N}(H)$', which reads: '*for any* $\mathbf{x} \in \mathbb{R}^p$, *if* $\mathbf{x} \in \mathcal{N}(GH)$ *then* $\mathbf{x} \in \mathcal{N}(H)$'.]

Pick any $\mathbf{u} \in \mathbb{R}^p$. Suppose $\mathbf{u} \in \mathcal{N}(GH)$.

[*Roughwork.* What to deduce? '$\mathbf{u} \in \mathcal{N}(H)$'. What does it read? '$H\mathbf{u} = \mathbf{0}_n$.'
How to reach '$H\mathbf{u} = \mathbf{0}_n$'? Find out what '$\mathbf{u} \in \mathcal{N}(GH)$' reads: it is '$(GH)\mathbf{u} = \mathbf{0}_m$'.]

Then by the definition of $\mathcal{N}(GH)$, we have $G(H\mathbf{u}) = (GH)\mathbf{u} = \mathbf{0}_m$.
Therefore, by the definition of $\mathcal{N}(G)$, we have $H\mathbf{u} \in \mathcal{N}(G)$.
Since $\mathcal{N}(G) = \{\mathbf{0}_n\}$, we have $H\mathbf{u} \in \{\mathbf{0}_n\}$. Then $H\mathbf{u} = \mathbf{0}_n$.
Therefore, by the definition of $\mathcal{N}(H)$, we have $\mathbf{u} \in \mathcal{N}(H)$.
It follows that $\mathcal{N}(GH) \subset \mathcal{N}(H)$.

12. **Example (j).**

Let S, T be subsets of $\mathbb{R}^n$, G be an $(m \times n)$-matrix with real entries, and H be an $(n \times p)$-matrix with real entries.

Define $S' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x}$ for some $\mathbf{x} \in S\}$, $T' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x}$ for some $\mathbf{x} \in T\}$.

Define $S^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u}$ for some $\mathbf{x} \in S\}$, $T^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u}$ for some $\mathbf{x} \in T\}$.

The statements below hold:

(1) *Suppose S is a subset of T. Then S' is a subset of T'.*

(2) *Suppose S is a subset of T. Then S^* is a subset of T^*.*

13. **Proofs of the statements in Example (j).**

(1) Suppose S is a subset of T.

[*Reminder.* We want to deduce, under the above assumption, that '*S' is a subset of T'* ', which reads: '*for any $\mathbf{y} \in \mathbb{R}^m$, if $\mathbf{y} \in S'$ then $\mathbf{y} \in T'$* '.
Recall what S' and T' are:
$S' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x}$ for some $\mathbf{x} \in S\}$,
$T' = \{\mathbf{y} \in \mathbb{R}^m : \mathbf{y} = G\mathbf{x}$ for some $\mathbf{x} \in T\}$, in which G is some fixed $(m \times n)$-matrix.]

Pick any object $\mathbf{y} \in \mathbb{R}^m$. Suppose $\mathbf{y} \in S'$.

[*Roughwork.* What to deduce? '$\mathbf{y} \in T'$ '. What does it read? 'Unwrap' '$\mathbf{y} \in T'$ ' to see what it is.
How to reach '$\mathbf{y} \in T'$ '? 'Unwrap' '$\mathbf{y} \in S'$ ' to see what may help us.]

Then by the definition of S', there exists some $\mathbf{x} \in S$ such that $\mathbf{y} = G\mathbf{x}$.
Note that $\mathbf{x} \in S$, and by assumption S is a subset of T. Then, by the definition of subset relations, $\mathbf{x} \in T$.
Therefore $\mathbf{x} \in T$ and $\mathbf{y} = G\mathbf{x}$ for the same $\mathbf{x}, \mathbf{y}$.
Hence, by the definition of T', we have $\mathbf{y} \in T'$.
It follows that $S' \subset T'$.

(2) Suppose S is a subset of T.

[*Reminder.* We want to deduce, under the above assumption, that that '*S^* is a subset of T^** ', which reads: '*for any $\mathbf{u} \in \mathbb{R}^p$, if $\mathbf{u} \in S^*$ then $\mathbf{u} \in T^*$* '.
Recall what S^* and T^* are:
$S^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u}$ for some $\mathbf{x} \in S\}$,
$T^* = \{\mathbf{u} \in \mathbb{R}^p : \mathbf{x} = H\mathbf{u}$ for some $\mathbf{x} \in T\}$, in which H is some fixed $(m \times p)$-matrix.]

Pick any object $\mathbf{u} \in \mathbb{R}^p$. Suppose $\mathbf{u} \in S^*$.

[*Roughwork.* What to deduce? '$\mathbf{u} \in T^*$ '. What does it read? 'Unwrap' '$\mathbf{u} \in T^*$ ' to see what it is.
How to reach '$\mathbf{u} \in T^*$ '? 'Unwrap' '$\mathbf{u} \in S^*$ ' to see what may help us.]

Then by the definition of S^*, there exists some $\mathbf{x} \in S$ such that $\mathbf{x} = H\mathbf{u}$.
Note that $\mathbf{x} \in S$, and by assumption S is a subset of T. Then, by the definition of subset relations, $\mathbf{x} \in T$.
Therefore $\mathbf{x} \in T$ and $\mathbf{x} = H\mathbf{u}$, for the same $\mathbf{x}, \mathbf{u}$.
Hence, by the definition of T^*, we have $\mathbf{u} \in T^*$.
It follows that $S^* \subset T^*$.