1. **Definition. (Intersection, union, complement.)**.

   Let A, B be sets.

   (a) The **intersection** of the sets A, B is defined to be the set $\{x \mid x \in A \text{ and } x \in B\}$. It is denoted by $A \cap B$.

   (b) The **union** of the sets A, B is defined to be the set $\{x \mid x \in A \text{ or } x \in B\}$. It is denoted by $A \cup B$.

   (c) The **complement of the set** B **in the set** A is defined to be the set $\{x \mid x \in A \text{ and } x \notin B\}$. It is denoted by $A \backslash B$.

   Remark. By definition:—

   - $x \in A \cap B$ iff ($x \in A$ and $x \in B$).
   - $x \in A \cup B$ iff ($x \in A$ or $x \in B$).
   - $x \in A \backslash B$ iff ($x \in A$ and $x \notin B$).

2. **Definition. (Symmetric difference.)**

   Let A, B be sets. The **symmetric difference** *of the sets A, B is defined to be the set $(A\backslash B) \cup (B\backslash A)$. It is denoted by $A \triangle B$.*

3. **Example (1).**

   (a) Suppose $A = \{0, 1\}$, $B = \{1, 2\}$. Then:—
   - $A \cap B = \{1\}$. (Its only element is 1.)
   - $A \cup B = \{0, 1, 2\}$. (Its elements are exactly $0, 1, 2$.)
   - $A \backslash B = \{0\}$. (Its only element is 0.)
   - $B \backslash A = \{2\}$. (Its only element is 2.)
   - $A \triangle B = \{0, 2\}$. (Its elements are exactly $0, 2$.)

   (b) Suppose $A = \{0, \{0\}\}$, $B = \{\{0\}, \{\{0\}\}\}$. Then:—
   - $A \cap B = \{\{0\}\}$. (Its only element is the object $\{0\}$.)
   - $A \cup B = \{0, \{0\}, \{\{0\}\}\}$. (Its elements are exactly the three objects $0, \{0\}, \{\{0\}\}$.)
   - $A \backslash B = \{0\}$. (Its only element is the object 0.)
   - $B \backslash A = \{\{\{0\}\}\}$. (Its only element is the object $\{\{0\}\}$.)
   - $A \triangle B = \{0, \{\{0\}\}\}$. (Its elements are exactly the two objects $0, \{\{0\}\}$.)

4. **Subset relation.**

   We have earlier introduced the notion of *subset relation*:

   - Given any two sets A, B, the set A is a subset of the set B exactly when every element of A belongs to B.

   We now give a formal definition for the notion of *subset relation*. Formal and clumsy though it looks, it is best to work with this definition in calculations or proofs, because its logical content has been spelt out explicitly.

   Definition. (Subset relation.)

   Let A, B be sets. We say A is a **subset** *of B if the statement (†) holds:*

   (†) *For any object x, [if ($x \in A$) then ($x \in B$)].*

   We write $A \subset B$ (or $B \supset A$).

5. **Definition. (Power set.)**

Let A be a set. The **power set** *of the set A is defined to be the set $\{S \mid S$ is a subset of $A\}$. It is denoted by $\mathfrak{P}(A)$.*

Remark. By definition, $S \in \mathfrak{P}(A)$ iff $S \subset A$.

6. **Example (2).**

$A = ?$	Elements of A?	Subsets of A? Elements of $\mathfrak{P}(A)$?	$\mathfrak{P}(A) = ?$
$\emptyset$	—	$\emptyset$	$\{\emptyset\}$
$\{0\}$	0	$\emptyset$, $\{0\}$	$\{\emptyset, \{0\}\}$
$\{0,1\}$	$0, 1$	$\emptyset$, $\{0\}$, $\{1\}$, $\{0,1\}$	$\{\emptyset, \{0\}, \{1\}, \{0,1\}\}$
$\{0,1,2\}$	$0, 1, 2$	$\emptyset$, $\{0\}$, $\{1\}$, $\{2\}$, $\{0,1\}$, $\{1,2\}$, $\{0,2\}$ $\{0,1,2\}$	$\{\emptyset, \{0\}, \{1\}, \{2\},$ $\{0,1\}, \{1,2\}, \{0,2\}, \{0,1,2\}\}$
$\{\emptyset\}$	$\emptyset$	$\emptyset, \{\emptyset\}$	$\{\emptyset, \{\emptyset\}\}$
$\mathfrak{P}(\emptyset)$	$\emptyset$	$\emptyset, \{\emptyset\}$	$\{\emptyset, \{\emptyset\}\}$
$\mathfrak{P}(\{\emptyset\})$	$\emptyset, \{\emptyset\}$	$\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}$	$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$
$\mathfrak{P}(\{\{\emptyset\}\})$	$\emptyset, \{\{\emptyset\}\}$	$\emptyset, \{\emptyset\}, \{\{\{\emptyset\}\}\}, \{\emptyset, \{\{\emptyset\}\}\}$	$\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\{\emptyset\}\}\}\}$

Remarks.

(1) $\emptyset$, $\{\emptyset\}$ are different objects.
$\emptyset$ is the empty set: it has no element.
$\{\emptyset\}$ is a non-empty set which has exactly one element, namely the object $\emptyset$. (A set with exactly one element is called a singleton.)

(2) In general, when A has exactly N elements, $\mathfrak{P}(A)$ will have exactly 2^N elements.

Example (3).

(a) What is $\mathfrak{P}(\mathfrak{P}(\emptyset))$ explicitly?
$\mathfrak{P}(\mathfrak{P}(\emptyset)) = \mathfrak{P}(\{\emptyset\}) = \{\emptyset, \{\emptyset\}\}$.

(b) What is $\mathfrak{P}(\mathfrak{P}(\{\emptyset\}))$ explicitly?
$\mathfrak{P}(\mathfrak{P}(\{\emptyset\})) = \mathfrak{P}(\{\emptyset, \{\emptyset\}\}) = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$.

(c) What is $\mathfrak{P}(\mathfrak{P}(\{\{\emptyset\}\}))$ explicitly?
$\mathfrak{P}(\mathfrak{P}(\{\{\emptyset\}\})) = \mathfrak{P}(\{\emptyset, \{\{\emptyset\}\}\}) = \{\emptyset, \{\emptyset\}, \{\{\{\emptyset\}\}\}, \{\emptyset, \{\{\emptyset\}\}\}\}$.

7. **Set equality.**

We have earlier introduced the notion of *subset equality*:

- Any two sets A, B are equal to each other as sets exactly when each of A, B contains as its elements every element of the other.

We now give a formal definition for the notion of *set equality* (which, alongside the notion of *subset relation*, will be used in the formulation of some theoretical results on basic set operations).

Definition. (Set equality.)

Let A, B be sets. We say A is **equal** *to B if both statements (†), (‡) hold:*

(†) *For any object x, [if $(x \in A)$ then $(x \in B)$].*

(‡) *For any object y, [if $(y \in B)$ then $(y \in A)$].*

We write $A = B$.

8. **Properties of subset relation, intersection, union, complement, symmetric difference, power set.**

Theorem (I).

The statements below hold:

(1) *Suppose A is a set. Then $A \subset A$.*

(2) *Suppose A, B are sets. Then $A = B$ iff $[(A \subset B)$ and $(B \subset A)]$.*

(3) *Let A, B, C be sets. Suppose $A \subset B$ and $B \subset C$. Then $A \subset C$.*

Theorem (II).

Let A, B be sets. The statements below hold:

(1) $A \cap B \subset A$.

(2) $A \cap B \subset B$.

(3) $A \backslash B \subset A$.

(4) $A \subset A \cup B$.

(5) $B \subset A \cup B$.

Theorem (III).

Let A be a set. The statements below hold:

(1) $\emptyset \subset A$.

(2) $A \cap \emptyset = \emptyset$.

(3) $A \cup \emptyset = A$.

(4) $A \backslash \emptyset = A$.

(5) $\emptyset \backslash A = \emptyset$.

(6) $A \triangle \emptyset = A$.

(7) $A \triangle A = \emptyset$.

Theorem (IV).

The statements below hold:

(1) *Let A, B, S be sets. Suppose $S \subset A$ and $S \subset B$. Then $S \subset A \cap B$.*

(2) *Let A, B, S be sets. Suppose $S \subset A$ or $S \subset B$. Then $S \subset A \cup B$.*

(3) *Let A, B, T be sets. Suppose $A \subset T$ and $B \subset T$. Then $A \cup B \subset T$.*

(4) *Let A, B, T be sets. Suppose $A \subset T$ or $B \subset T$. Then $A \cap B \subset T$.*

Theorem (V). *Let A, B, C be sets. The statements below hold:*

(1) $A \cap A = A$.

(2) $A \cap B = B \cap A$.

(3) $(A \cap B) \cap C = A \cap (B \cap C)$.

(4) $(A \cap B) \cup C = (A \cup C) \cap (B \cup C)$.

(5) $(A \cap B) \backslash C = (A \backslash C) \cap (B \backslash C)$.

(6) $A \backslash (B \cap C) = (A \backslash B) \cup (A \backslash C)$.

(1') $A \cup A = A$.

(2') $A \cup B = B \cup A$.

(3') $(A \cup B) \cup C = A \cup (B \cup C)$.

(4') $(A \cup B) \cap C = (A \cap C) \cup (B \cap C)$.

(5') $(A \cup B) \backslash C = (A \backslash C) \cup (B \backslash C)$.

(6') $A \backslash (B \cup C) = (A \backslash B) \cap (A \backslash C)$.

(7) $A \triangle B = (A \cup B) \backslash (A \cap B)$.

(8) $A \triangle B = B \triangle A$.

(9) $(A \triangle B) \triangle C = A \triangle (B \triangle C)$.

Theorem ($\dagger_1$).

Let A, B be sets. The statements below hold:

(1) $\emptyset, A \in \mathfrak{P}(A)$. *Moreover,* $\mathfrak{P}(A) \neq \emptyset$.

(2) (a) *Suppose $A \subset B$. Then $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.*

(b) *Suppose $\mathfrak{P}(A) \subset \mathfrak{P}(B)$. Then $A \subset B$.*

(c) *$A \subset B$ iff $\mathfrak{P}(A) \subset \mathfrak{P}(B)$.*

(3) $\mathfrak{P}(A \cap B) = \mathfrak{P}(A) \cap \mathfrak{P}(B)$.

(4) $\mathfrak{P}(A) \cup \mathfrak{P}(B) \subset \mathfrak{P}(A \cup B)$.

Theorem ($\dagger_2$).

Let A, B be sets. The statements below hold:

(1) *Suppose $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$. Then ($A \subset B$ or $B \subset A$).*

(2) *Suppose ($A \subset B$ or $B \subset A$). Then $\mathfrak{P}(A \cup B) \subset \mathfrak{P}(A) \cup \mathfrak{P}(B)$.*

(3) $\mathfrak{P}(A \cup B) = \mathfrak{P}(A) \cup \mathfrak{P}(B)$ *iff ($A \subset B$ or $B \subset A$).*