

1. Many a set cannot be presented as a list, because it is not ‘small’. Even though a set can be presented as a list, for one reason or another it may be undesirable to do so.

Illustrations.

- (a) Consider the collection ‘0, 1, 4, 9, 16, 25, 36, ...’.

Is it apparent that it refers to the collection of all square integers?

But why can’t it be understood as the collection of 0, 1, 4, 9, 16, 25 and the integers no less than 36?

- (b) Consider the collection ‘1, 2, 3, 4, ...; $\frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \dots$; $\frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \dots$; ...’

Is it apparent that it refers to the collection of all positive rational numbers? Or is it not?

Can you conceive a better list than this one?

Or is it desirable to describe the collection of all positive rational numbers in this way?

2. When it is impossible or undesirable to present a set by exhaustively listing every element of the set, we may try the **Method of Specification**:

We pinpoint the elements of the set concerned by writing down an appropriate ‘*selection criterion*’ which instructs ourselves exactly which objects are to be collected (and which not).

Formally speaking, to construct a set through this method, we write down an appropriate predicate, say, $P(x)$, with one variable x , for which:—

- an object to be ‘collected’ as an element of the set concerned will convert $P(x)$ into a true statement upon its being substituted into x , while
- an object to be ‘discarded’ will convert $P(x)$ into a false statement upon its being substituted into x .

3. **Mathematical statements and predicates.**

- A **mathematical statement** is a sentence with mathematical content (or several inter-related sentences which can be condensed into one through the appropriate use of clauses), for which it is meaningful to say it is true or it is false.

- A **predicate with variables** x, y, z, \dots is a statement ‘modulo’ the ambiguity of possibly one or several variables x, y, z, \dots .

Provided we have specified x, y, z, \dots in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

4. **Two forms of Method of Specification.**

Suppose A is a set, and $P(x)$ is a predicate with variable x .

- (a) $\{x \mid P(x)\}$ refers to the set (if it is indeed a set) which contains exactly every object x for which the statement $P(x)$ is true.
- (b) $\{x \in A : P(x)\}$ refers to the set which contains exactly every object x which is an element of the given set A and for which the statement $P(x)$ is true. By definition it is a subset of A .

Reminder. The symbol ‘ x ’ in the expressions ‘ $\{x \mid P(x)\}$ ’, ‘ $\{x \in A : P(x)\}$ ’ are dummies.

- (a) $\{x \mid P(x)\}$, $\{u \mid P(u)\}$, $\{c \mid P(c)\}$ all refer to the same set.
- (b) $\{x \in A : P(x)\}$, $\{u \in A : P(u)\}$, $\{c \in A : P(c)\}$ all refer to the same set.

Another reminder. Always remember (how this construction is used in practice):—

- (a) Suppose S stands for the set $\{x \mid P(x)\}$, and u is an object.
Then $u \in S$ iff $P(u)$ is a true statement.
- (b) Suppose T stands for the set $\{x \in A : P(x)\}$, and u is an object.
Then $u \in T$ iff ($u \in A$ and $P(u)$ is a true statement).

5. **Example (1).**

- (a) The predicate ‘ $x = 1$ or $x = 2$ or $x = 3$ ’ is converted into a true statement exactly when any one of 1, 2, 3 into the variable x .
Hence $\{x \mid x = 1 \text{ or } x = 2 \text{ or } x = 3\} = \{1, 2, 3\}$ as sets.
- (b) The predicate ‘ $x^2 - 3x + 2 = 0$ ’ is converted into a true statement exactly when any one of 1, 2 is substituted into the variable x .
Hence $\{x \mid x^2 - 3x + 2 = 0\} = \{1, 2\}$ as sets.
- (c) The predicate ‘ $x^2 + 4 = 0$ ’ is converted into a true statement exactly when any one of $2i, -2i$ is substituted into the variable x .
Hence:
i. $\{x \mid x^2 + 4 = 0\} = \{2i, -2i\}$ as sets.
ii. $\{x \in \mathbb{C} : x^2 + 4 = 0\} = \{2i, -2i\}$ as sets.
iii. $\{x \in \mathbb{R} : x^2 + 4 = 0\} = \emptyset$.
- (d) The predicate ‘ $(x^2 + 1)(x^2 - 2) = 0$ ’ is converted into a true statement exactly when any one of $\sqrt{2}, -\sqrt{2}, i, -i$ is substituted into the variable x .
Hence:
i. $\{x \mid (x^2 + 1)(x^2 - 2) = 0\} = \{\sqrt{2}, -\sqrt{2}, i, -i\}$ as sets.
ii. $\{x \in \mathbb{C} : (x^2 + 1)(x^2 - 2) = 0\} = \{\sqrt{2}, -\sqrt{2}, i, -i\}$ as sets.
iii. $\{x \in \mathbb{R} : (x^2 + 1)(x^2 - 2) = 0\} = \{\sqrt{2}, -\sqrt{2}\}$ as sets.
iv. $\{x \in \mathbb{Q} : (x^2 + 1)(x^2 - 2) = 0\} = \emptyset$ as sets.
- (e) The predicate ‘ $x^2 - 3x + 2 < 0$ ’ is converted into a true statement exactly when any one real number strictly between 1 and 2 is substituted into the variable x .
Hence $\{x \in \mathbb{R} : x^2 - 3x + 2 < 0\} = \{x \in \mathbb{R} : 1 < x < 2\}$ as sets.

6. Solution sets, as sets constructed by the Method of Specification.

Every equation/inequality with unknown so-and-so in the set blah-blah-blah can be thought of as a predicate with one variable. That variable is usually denoted by the same so-and-so as well.

An object is called a **solution** of that equation/inequality exactly when, upon the object being substituted into the equation/inequality concerned, a true statement is obtained.

By the phrase *solution set* for such an equation/inequality, we mean the subset of the set blah-blah-blah constructed through the Method of Specification with the equation/inequality concerned being used as the ‘selection criterion’.

Hence by definition, an object belong to the solution set for such an equation/inequality exactly when the object is a solution of that equation/inequality.

Illustrations.

- (a) The equation $x^2 - 3x + 2 = 0$ with real unknown x is a predicate with variable x .
The solution set for this equation is the set $\{x \in \mathbb{R} : x^2 - 3x + 2 = 0\}$.
- (b) The inequality $x^2 - 3x + 2 < 0$ with real unknown x is a predicate with variable x .
The solution set is the set $\{x \in \mathbb{R} : x^2 - 3x + 2 < 0\}$.
- (c) The equation $x^2 + 4 = 0$ with complex unknown x is a predicate with variable x .
The solution set for this equation is the set $\{x \in \mathbb{C} : x^2 + 4 = 0\}$.
- (d) The equation $\sin(x) = \frac{1}{2}$ with real unknown x is a predicate with variable x .
The solution set for this equation is the set $\left\{x \in \mathbb{R} : \sin(x) = \frac{1}{2}\right\}$.
- (e) Define $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$.
The equation $A\mathbf{x} = \mathbf{b}$ with unknown vector \mathbf{x} in \mathbb{R}^3 is a predicate with variable \mathbf{x} .
The solution set for this equation is the set $\{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{b}\}$.
(Here \mathbb{R}^2 is regarded as the set of all vectors with two real entries.)

(f) Let A be an $(m \times n)$ -matrix with real entries.

The equation $A\mathbf{x} = \mathbf{0}$ with unknown vector \mathbf{x} in \mathbb{R}^n is a predicate with variable \mathbf{x} .

The solution set for this equation is the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$.

As introduced in your *linear algebra* course, this set is called the **null space** of A .

7. Definition. (Intervals as sets constructed with the Method of Specification.)

The set S is said to be an **interval** in \mathbb{R} if any one of the statements below hold:

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|---|---|
| (a) $S = \emptyset$. | (f) $S = \{x \in \mathbb{R} : a < x < b\}$ for some $a, b \in \mathbb{R}$. |
| (b) $S = \{x \in \mathbb{R} : a < x\}$ for some $a \in \mathbb{R}$. | (g) $S = \{x \in \mathbb{R} : a \leq x < b\}$ for some $a, b \in \mathbb{R}$. |
| (c) $S = \{x \in \mathbb{R} : a \leq x\}$ for some $a \in \mathbb{R}$. | (h) $S = \{x \in \mathbb{R} : a < x \leq b\}$ for some $a, b \in \mathbb{R}$. |
| (d) $S = \{x \in \mathbb{R} : x < b\}$ for some $b \in \mathbb{R}$. | (i) $S = \{x \in \mathbb{R} : a \leq x \leq b\}$ for some $a, b \in \mathbb{R}$. |
| (e) $S = \{x \in \mathbb{R} : x \leq b\}$ for some $b \in \mathbb{R}$. | (j) $S = \mathbb{R}$. |

Remark on notations and terminologies. We write:

- | | |
|--|---|
| • $(a, +\infty) = \{x \in \mathbb{R} : a < x\}$. | • $(a, b) = \{x \in \mathbb{R} : a < x < b\}$. |
| • $[a, +\infty) = \{x \in \mathbb{R} : a \leq x\}$. | • $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$. |
| • $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$. | • $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$. |
| • $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$. | • $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$. |

Each of the numbers a, b is called an endpoint of the interval concerned. If the interval concerned contains all of its endpoints as its elements, it is said to be a **closed interval**. If it contains none, it is said to be an **open interval**. If the interval is bounded in \mathbb{R} , it is said to be a **bounded interval**. If it is not bounded in \mathbb{R} , it is said to be an **unbounded interval**.

We agree to say that \emptyset is both an open interval and a closed interval. We also agree to say that \mathbb{R} is both an open interval and a closed interval.

8. Example (2).

Very often a predicate, say, $P(x)$, with variable x , used as the ‘selection criterion’ in the application of the Method of Specification is of the form

‘there exists some blah-blah-blah such that bleh-bleh-bleh’

in which the variable x sits inside ‘bleh-bleh-bleh’.

(a) The predicate (with variable x) given by

‘there exists some $n \in \mathbb{N}$ such that $x = n^2$ ’

is converted into a true statement exactly when any one square integers $(0, 1, 4, 9, 16, \dots)$ is substituted into the variable x .

Hence

$$\{x \mid \text{there exists some } n \in \mathbb{N} \text{ such that } x = n^2\}$$

is the same as the set of all square integers ‘ $\{0, 1, 4, 9, 16, \dots\}$ ’.

The use of the Method of Specification spares us the trouble of having to clarifying what we mean by the dot-dot-dot’s in ‘ $0, 1, 4, 9, 16, \dots$ ’.

Remark on short-hand. When we allow the predicate

‘there exists some $n \in \mathbb{N}$ such that $x = n^2$ ’

to be presented as

‘ $x = n^2$ for some $n \in \mathbb{N}$ ’,

we may present the set

$$\{x \mid \text{there exists some } n \in \mathbb{N} \text{ such that } x = n^2\}$$

as

$$\{x \mid x = n^2 \text{ for some } n \in \mathbb{N}\}.$$

We may ‘abbreviate’ the last expression as

$$\{n^2 \mid n \in \mathbf{N}\}.$$

This ‘short-hand’ is visually appealing as it suggests that the set concerned is constructed by collecting those and only those ‘numbers of the form n^2 ’ obtained ‘when n runs through all possible natural numbers’.

- (b) The predicate (with variable x) given by

‘there exists some $n \in \mathbf{Z}$ such that $x = 2n$ ’

is converted into a true statement exactly when any one even integers ($0, \pm 2, \pm 4, \pm 6, \dots$) is substituted into the variable x .

Hence:

- i. $\{x \mid \text{there exists some } n \in \mathbf{Z} \text{ such that } x = 2n\}$ is the same as the set of all even integers ‘ $\{\dots, -6, -4, -2, 0, 2, 4, 6, \dots\}$ ’.
- ii. $\{x \in \mathbf{N} : \text{there exists some } n \in \mathbf{Z} \text{ such that } x = 2n\}$ is the same as the set of all non-negative integers ‘ $\{0, 2, 4, 6, 8, \dots\}$ ’.

- (c) The predicate (with variable x) given by

‘there exists some $n \in \mathbf{Z}$ such that $x = n^3$ ’

is converted into a true statement exactly when any one even integers ($0, \pm 1, \pm 8, \pm 27, \dots$) is substituted into the variable x .

Hence:

- i. $\{x \mid \text{there exists some } n \in \mathbf{Z} \text{ such that } x = n^3\}$ is the same as the set of all cube integers ‘ $\{\dots, -27, -8, -1, 0, 1, 8, 27, \dots\}$ ’.
- ii. $\{x \in \mathbf{N} : \text{there exists some } n \in \mathbf{Z} \text{ such that } x = n^3\}$ is the same as the set of all non-negative cube integers ‘ $\{0, 1, 8, 27, 64, \dots\}$ ’.

- (d) The predicate (with variable x) given by

‘there exist some $m, n \in \mathbf{N}$ such that $x = 2^m \cdot 3^n$ ’

is converted into a true statement exactly when any number which can be factorized as a product of 2’s and 3’s is substituted into the variable x .

Hence $\{x \mid \text{there exist some } m, n \in \mathbf{N} \text{ such that } x = 2^m \cdot 3^n\}$ is the same as the set ‘ $\{1, 2, 3, 4, 6, 8, 9, 12, 18, 27, \dots\}$ ’.

- (e) The predicate (with variable z) given by

‘there exist some $n \in \mathbf{N}$ such that $|z| = n$ ’

is converted into a true statement exactly when any complex number whose modulus is a natural number is substituted into the variable z .

Hence $\{z \in \mathbf{C} : \text{there exist some } n \in \mathbf{N} \text{ such that } |z| = n\}$ is the same as the set $\{z \in \mathbf{C} : |z| \in \mathbf{N}\}$.

It is the ‘union’ of all circles with centre at 0 and with radius being a natural number.

- (f) Let H be a $(p \times q)$ -matrix.

The predicate (with variable \mathbf{y}) given by

‘there exist some $\mathbf{t} \in \mathbb{R}^q$ such that $\mathbf{y} = H\mathbf{t}$ ’

is converted into a true statement exactly when a vector in \mathbb{R}^p which ‘can be expressed as $H\mathbf{t}$ (for some appropriate vector \mathbf{t})’ is substituted into the variable \mathbf{y} .

The set $\{\mathbf{y} \in \mathbb{R}^p : \text{there exist some } \mathbf{t} \in \mathbb{R}^q \text{ such that } \mathbf{y} = H\mathbf{t}\}$ is the **column space** of the matrix H , as introduced in your *linear algebra* course.

Short-hand. We write this set as

$$\{\mathbf{y} \in \mathbb{R}^p : \mathbf{y} = H\mathbf{t} \text{ for some } \mathbf{t} \in \mathbb{R}^q\},$$

or even as

$$\{H\mathbf{t} \mid \mathbf{t} \in \mathbb{R}^q\}.$$

This ‘short-hand’ is visually appealing as it suggests that the set concerned is constructed by collecting those and only those ‘vectors of the form $H\mathbf{t}$ ’ obtained ‘when \mathbf{t} runs through all possible elements of \mathbb{R}^q ’.

- (g) Let $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ be vectors in \mathbb{R}^n .

The predicate (with variable \mathbf{y}) given by

‘there exist some $\alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R}$ such that $\mathbf{y} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k$ ’

is converted into a true statement exactly when a vector in \mathbb{R}^n which ‘can be expressed as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ ’ is substituted into the variable \mathbf{y} .

The set $\left\{ \mathbf{y} \in \mathbb{R}^n : \begin{array}{l} \text{there exist some } \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R} \\ \text{such that } \mathbf{y} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k \end{array} \right\}$ is the **span** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$, as introduced in your *linear algebra* course.

Short-hand. We write this set as

$$\left\{ \mathbf{y} \in \mathbb{R}^n : \begin{array}{l} \mathbf{y} = \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k \\ \text{for some } \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R} \end{array} \right\},$$

or even as

$$\{ \alpha_1 \mathbf{u}_1 + \alpha_2 \mathbf{u}_2 + \dots + \alpha_k \mathbf{u}_k \mid \alpha_1, \alpha_2, \dots, \alpha_k \in \mathbb{R} \}.$$

This ‘short-hand’ is visually appealing as it suggests that the set concerned is constructed by collecting all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \dots, \mathbf{u}_k$ in \mathbb{R}^n .

9. ‘Solving an equation/inequality’, understood as finding an ‘explicit presentation’ of its solution set.

To ‘solve an equation/inequality’ is the same as to determine all its solutions and find a way to present them explicitly. Very often such an equation/inequality has (infinitely) many solutions, and no one specific solution is privileged over another.

This prompts us to present the collection of all solutions for such an equation/inequality as a set constructed by the method of specification, in which the ‘selection criterion, used in the application of this method gives an explicit description of all individual solutions of the equation/inequality.

Illustrations.

- (a) The solutions of the equation $x^2 - 3x + 2 = 0$ with real unknown x are exactly 1, 2.

We may present the content of the above statement in the form of the set equality

$$\{x \in \mathbb{R} : x^2 - 3x + 2 = 0\} = \{1, 2\}.$$

The left-hand side is the solution set of the equation $x^2 - 3x + 2 = 0$ with real unknown x .

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

- (b) The solutions of the equation $x^2 + 4 = 0$ with complex unknown x are exactly $2i, -2i$.

We may present the content of the above statement in the form of the set equality

$$\{x \in \mathbb{C} : x^2 + 4 = 0\} = \{2i, -2i\}.$$

The left-hand side is the solution set of the equation $x^2 + 4 = 0$ with complex unknown x .

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

- (c) There is no solution for the equation $x^2 + 4 = 0$ with real unknown x .

We may present the content of the above statement in the form of the set equality

$$\{x \in \mathbb{R} : x^2 + 4 = 0\} = \emptyset.$$

The left-hand side is the solution set of the equation $x^2 + 4 = 0$ with real unknown x .

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

- (d) The solutions of the inequality $x^2 - 3x + 2 < 0$ with real unknown x are exactly the real numbers strictly between 1 and 2.

We may present the content of the above statement in the form of the set equality

$$\{x \in \mathbb{R} : x^2 - 3x + 2 < 0\} = \{x \in \mathbb{R} : 1 < x < 2\}.$$

The left-hand side is the solution set of the equation $x^2 - 3x + 2 < 0$ with real unknown x .

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

- (e) Suppose α is a real number. Then α is a solution $\sin(x) = \frac{1}{2}$ with real unknown x iff there exists some $n \in \mathbb{Z}$ such that $\alpha = n\pi + (-1)^n \cdot \frac{\pi}{6}$.

We may present the content of the above statement in the form of the set equality

$$\left\{ x \in \mathbb{R} : \sin(x) = \frac{1}{2} \right\} = \left\{ x \in \mathbb{R} : \begin{array}{l} \text{There exist some } n \in \mathbb{Z} \text{ such that} \\ n\pi + (-1)^n \cdot \frac{\pi}{6} \end{array} \right\}.$$

The left-hand side is the solution set of the equation $\sin(x) = \frac{1}{2}$ with real unknown x .

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

- (f) Define $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$.

Suppose \mathbf{v} is a vector in \mathbb{R}^3 . Then \mathbf{v} is a solution of the equation $A\mathbf{x} = \mathbf{b}$ with unknown vector \mathbf{x} in \mathbb{R}^3 iff

there exists some $t \in \mathbb{R}$ such that $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

We may present the content of the above statement in the form of the set equality

$$\left\{ \mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{b} \right\} = \left\{ \mathbf{x} \in \mathbb{R}^3 : \text{There exists some } t \in \mathbb{R} \text{ such that } \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}.$$

The left-hand side is the solution set of the equation $A\mathbf{x} = \mathbf{b}$ with real vector unknown \mathbf{x} .

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

10. Example (3).

- (a) The predicate ' $x \neq x$ ' yields a false statement, no matter which object is substituted into it. For this reason, the set $\{x \mid x \neq x\}$ is the empty set. Also, for any set A , the set $\{x \in A : x \neq x\}$ is the empty set.
- (b) Consider the predicate '*for any* $n \in \mathbb{N}$, $x = n^2$ ', in which the variable is x . This predicate yields a false statement no matter which object is substituted into x . Hence the set $\{x \mid \text{for any } n \in \mathbb{N}, x = n^2\}$ is the empty set.

Remark. Contrast this set with $\{x \mid \text{there exists some } n \in \mathbb{N} \text{ such that } x = n^2\}$. The latter is the set of all square integers.