1. Many a set cannot be presented as a list, because it is not 'small'. Even though a set can be presented as a list, for one reason or another it may be undesirable to do so.

Illustrations.

(a) Consider the collection '$0, 1, 4, 9, 16, 25, 36, \cdots$'.

Is it apparent that it refers to the collection of all square integers?

But why can't it be understood as the collection of $0, 1, 4, 9, 16, 25$ and the integers no less than 36?

(b) Consider the collection '$1, 2, 3, 4, \cdots; \frac{1}{2}, \frac{3}{2}, \frac{5}{2}, \frac{7}{2}, \cdots; \frac{1}{3}, \frac{2}{3}, \frac{4}{3}, \frac{5}{3}, \cdots; ...$'

Is it apparent that it refers to the collection of all positive rational numbers? Or is it not?

Can you conceive a better list than this one?

Or is it desirable to describe the collection of all positive rational numbers in this way?

2. When it is impossible or undesirable to present a set by exhaustively listing every element of the set, we may try the **Method of Specification**:

   We pinpoint the elements of the set concerned by writing down an appropriate '*selection criterion*' which instructs ourselves exactly which objects are to be collected (and which not).

   Formally speaking, to construct a set through this method, we write down an appropriate predicate, say, $P(x)$, with one variable x, for which:—

   - an object to be 'collected' as an element of the set concerned will convert $P(x)$ into a true statement upon its being substituted into x, while
   - an object to be 'discarded' will convert $P(x)$ into a false statement upon its being substituted into x.

3. **Mathematical statements and predicates**.

   - *A* **mathematical statement** *is a sentence with mathematical content (or several inter-related sentences which can be condensed into one through the appropriate use of clauses), for which it is meaningful to say it is true or it is false.*
   - *A* **predicate with variables** $x, y, z, \cdots$ *is a statement 'modulo' the ambiguity of possibly one or several variables* $x, y, z, \cdots$.

     Provided we have specified $x, y, z, \cdots$ in such a predicate, it becomes a statement, for which it makes sense to say it is true or false.

Examples of statements.

(Some of them are true statements. Some of them are false statements.)

(a) $1+1=2$.

(b) $1+1>3$.

(c) $\sqrt{2}$ is an irrational number.

(d) 6 is divisible by 8.

(e) There exists some natural number n such that n is divisible by 0.

(f) For any real number x, for any real number y, if $x<y$ then $x^2<y^2$.

(g) For any real number x, there exists some rational number y such that $|x-y|^2<|x-y|+1$.

(h) There exists some natural number n such that for any real number x, $n\leq|x|$.

Examples of predicates with some variables.

(Once the variables concerned are replaced by concrete objects, the respective predicates become statements.)

(a) $x+1=2$. (a′) $x+y=2$. (a″) $x+1=y$.

(b) $x+y>3$. (b′) $x+y>z$.

(c) $\sqrt{x}$ is an irrational number.

(d) x is divisible by y.

(e) n is divisible by 0.

(e′) There exists some natural number n such that n is divisible by x.

(f) For any real number y, if $x<y$ then $x^2<y^2$.

(f′) If $x<y$ then $x^2<y^2$.

(g) There exists some rational number y such that $|x-y|^2<|x-y|+1$.

(g′) $|x-y|^2<|x-y|+1$.

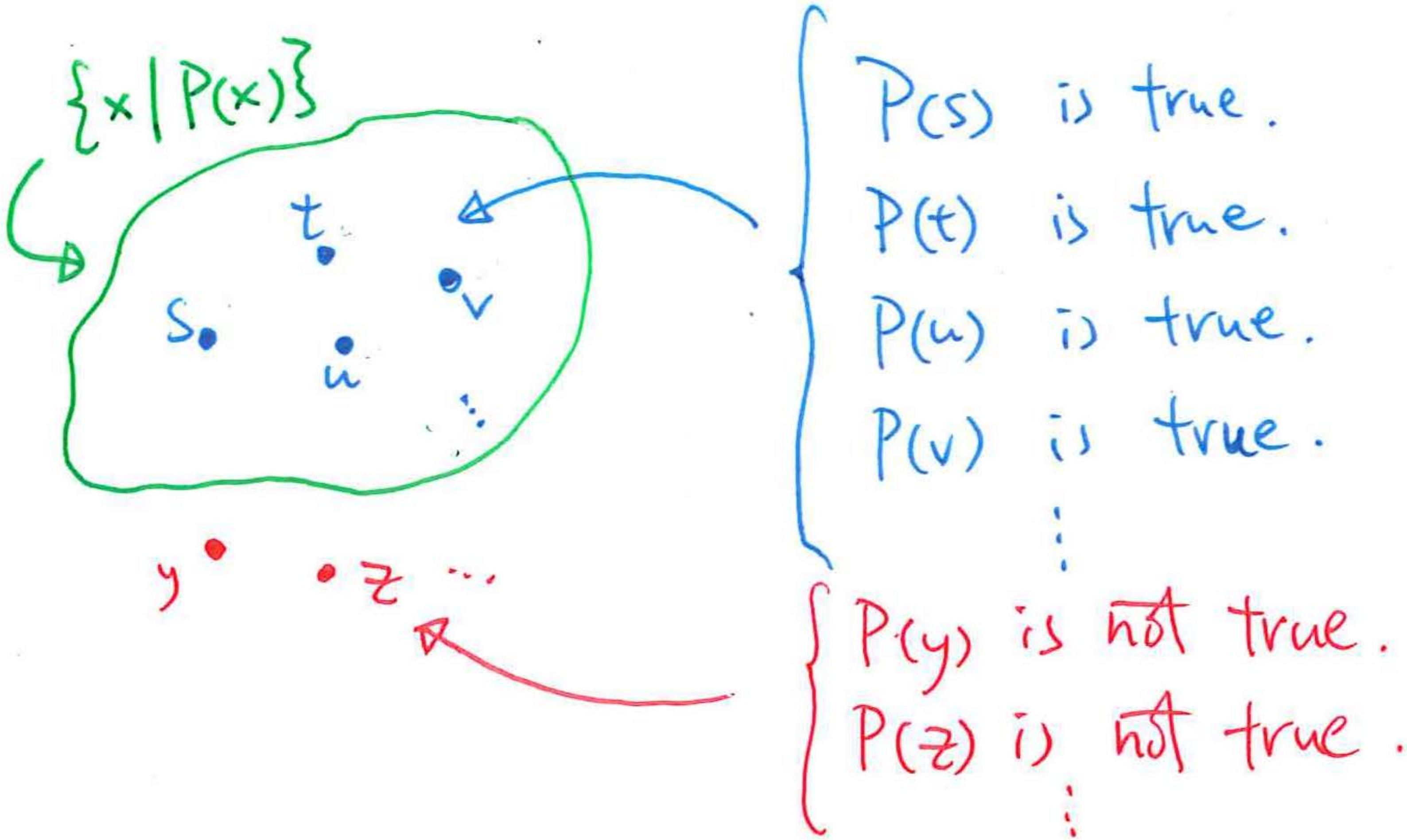
(g″) For any real number x, there exists some rational number y such that $|x-y|^2<|x-y|+z$.

4. **Two forms of Method of Specification.**

Suppose A is a set, and $P(x)$ is a predicate with variable x.

(a) $\{x \mid P(x)\}$ refers to the set (if it is indeed a set) which contains exactly every object x

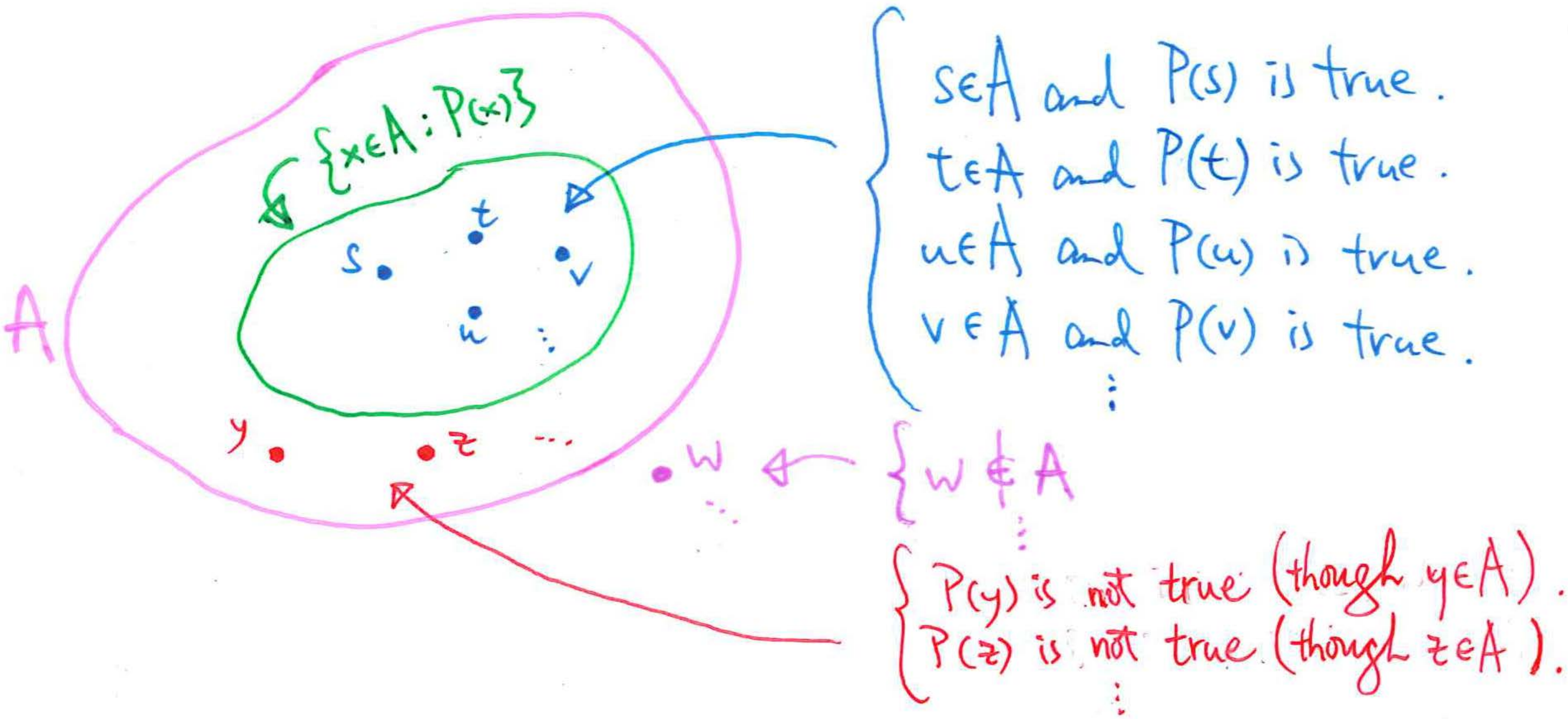
* for which the statement $P(x)$ is true.



(b) $\{x \in A : P(x)\}$ refers to the set which contains exactly every object x

- $*$ which is an element of the given set A and
- $*$ for which the statement $P(x)$ is true.

By definition it is a subset of A.



Reminder.

The symbol 'x' in the expressions

$$`\{x \mid P(x)\}', `\{x \in A : P(x)\}'$$

are dummies.

(a) $\{x \mid P(x)\}$, $\{u \mid P(u)\}$, $\{c \mid P(c)\}$ all refer to the same set.

(b) $\{x \in A : P(x)\}$, $\{u \in A : P(u)\}$, $\{c \in A : P(c)\}$ all refer to the same set.

Another reminder.

Always remember (how this construction is used in practice):—

(a) Suppose S stands for the set $\{x \mid P(x)\}$, and u is an object.

Then

$$u \in S \text{ iff } P(u) \text{ is a true statement.}$$

(b) Suppose T stands for the set $\{x \in A : P(x)\}$, and u is an object.

Then

$$u \in T \text{ iff } (u \in A \text{ and } P(u) \text{ is a true statement}).$$

5. **Example (1).**

(a) The predicate

$$\text{`}x = 1 \textit{ or } x = 2 \textit{ or } x = 3\text{'}$$

is converted into a true statement exactly when:

any one of $1, 2, 3$ into the variable x.

Hence

$$\{x \mid x = 1 \text{ or } x = 2 \text{ or } x = 3\} = \{1, 2, 3\} \quad \text{as sets.}$$

(b) The predicate

$$\text{`}x^2 - 3x + 2 = 0\text{'}$$

is converted into a true statement exactly when:

any one of $1, 2$ is substituted into the variable x.

Hence

$$\{x \mid x^2 - 3x + 2 = 0\} = \{1, 2\} \quad \text{as sets.}$$

(c) The predicate

$$`x^2 + 4 = 0'$$

is converted into a true statement exactly when

any one of $2i, -2i$ is substituted into the variable x.

Hence:

i. $\{x \mid x^2 + 4 = 0\} = \{2i, -2i\}$ as sets.

ii. $\{x \in \mathbb{C} : x^2 + 4 = 0\} = \{2i, -2i\}$ as sets.

iii. $\{x \in \mathbb{R} : x^2 + 4 = 0\} = \emptyset$.

(d) The predicate

$$\text{`}(x^2+1)(x^2-2)=0\text{'}$$

is converted into a true statement exactly when:

any one of $\sqrt{2}, -\sqrt{2}, i, -i$ is substituted into the variable x.

Hence:

i. $\{x \mid (x^2+1)(x^2-2)=0\} = \{\sqrt{2}, -\sqrt{2}, i, -i\}$ as sets.

ii. $\{x \in \mathbb{C} : (x^2+1)(x^2-2)=0\} = \{\sqrt{2}, -\sqrt{2}, i, -i\}$ as sets.

iii. $\{x \in \mathbb{R} : (x^2+1)(x^2-2)=0\} = \{\sqrt{2}, -\sqrt{2}\}$ as sets.

iv. $\{x \in \mathbb{Q} : (x^2+1)(x^2-2)=0\} = \emptyset$ as sets.

(e) The predicate

$$\text{`}x^2-3x+2<0\text{'}$$

is converted into a true statement exactly when:

any one real number strictly between 1 and 2 is substituted into the variable x.

Hence

$$\{x \in \mathbb{R} : x^2-3x+2<0\} = \{x \in \mathbb{R} : 1<x<2\} \quad \text{as sets.}$$

6. **Solution sets, as sets constructed by the Method of Specification.**

Every equation/inequality with unknown so-and-so in the set blah-blah-blah can be thought of as a predicate with one variable. That variable is usually denoted by the same so-and-so as well.

An object is called a **solution** of that equation/inequality exactly when:—

> upon the object being substituted into the equation/inequality concerned, a true statement is obtained.

By the phrase *solution set* for such an equation/inequality, we mean the subset of the set blah-blah-blah constructed through the Method of Specification with the equation/inequality concerned being used as the 'selection criterion'.

Hence by definition, an object belong to the solution set for such an equation/inequality exactly when:

> the object is a solution of that equation/inequality.

Illustrations.

(a) The equation $x^2 - 3x + 2 = 0$ with real unknown x is a predicate with variable x.

The solution set for this equation is the set $\{x \in \mathbb{R} : x^2 - 3x + 2 = 0\}$.

(b) The inequality $x^2 - 3x + 2 < 0$ with real unknown x is a predicate with variable x.

The solution set is the set $\{x \in \mathbb{R} : x^2 - 3x + 2 < 0\}$.

(c) The equation $x^2 + 4 = 0$ with complex unknown x is a predicate with variable x.

The solution set for this equation is the set $\{x \in \mathbb{C} : x^2 + 4 = 0\}$.

(d) The equation $\sin(x) = \frac{1}{2}$ with real unknown x is a predicate with variable x.

The solution set for this equation is the set $\left\{x \in \mathbb{R} : \sin(x) = \frac{1}{2}\right\}$.

(e) Define $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$.

The equation $A\mathbf{x} = \mathbf{b}$ with unknown vector $\mathbf{x}$ in $\mathbb{R}^3$ is a predicate with variable $\mathbf{x}$.

The solution set for this equation is the set $\{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{b}\}$.
(Here $\mathbb{R}^2$ is regarded as the set of all vectors with two real entries.)

(f) Let A be an $(m \times n)$-matrix with real entries.

The equation $A\mathbf{x} = \mathbf{0}$ with unknown vector $\mathbf{x}$ in $\mathbb{R}^n$ is a predicate with variable $\mathbf{x}$.

The solution set for this equation is the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$.

In your *linear algebra* course, this set is called the **null space** of A.

7. **Definition. (Intervals as sets constructed with the Method of Specification.)**

Let S be a subset of $\mathbb{R}$. The set S is said to be an **interval in** $\mathbb{R}$ *if any one of the statements below hold:*

(a) $S = \emptyset$.

(b) $S = \{x \in \mathbb{R} : a < x\}$ *for some* $a \in \mathbb{R}$.

(c) $S = \{x \in \mathbb{R} : a \leq x\}$ *for some* $a \in \mathbb{R}$.

(d) $S = \{x \in \mathbb{R} : x < b\}$ *for some* $b \in \mathbb{R}$.

(e) $S = \{x \in \mathbb{R} : x \leq b\}$ *for some* $b \in \mathbb{R}$.

(f) $S = \{x \in \mathbb{R} : a < x < b\}$ *for some* $a, b \in \mathbb{R}$.

(g) $S = \{x \in \mathbb{R} : a \leq x < b\}$ *for some* $a, b \in \mathbb{R}$.

(h) $S = \{x \in \mathbb{R} : a < x \leq b\}$ *for some* $a, b \in \mathbb{R}$.

(i) $S = \{x \in \mathbb{R} : a \leq x \leq b\}$ *for some* $a, b \in \mathbb{R}$.

(j) $S = \mathbb{R}$.

Remark on notations and terminologies. We write:

- $(a, +\infty) = \{x \in \mathbb{R} : a < x\}$.
- $[a, +\infty) = \{x \in \mathbb{R} : a \leq x\}$.
- $(-\infty, b) = \{x \in \mathbb{R} : x < b\}$.
- $(-\infty, b] = \{x \in \mathbb{R} : x \leq b\}$.
- $(a, b) = \{x \in \mathbb{R} : a < x < b\}$.
- $[a, b) = \{x \in \mathbb{R} : a \leq x < b\}$.
- $(a, b] = \{x \in \mathbb{R} : a < x \leq b\}$.
- $[a, b] = \{x \in \mathbb{R} : a \leq x \leq b\}$.

Each of the numbers a, b is called an endpoint of the interval concerned.

If the interval concerned contains all of its endpoints as its elements, it is said to be a **closed interval**.

If it contains none, it is said to be an **open interval**.

If the interval is bounded in $\mathbb{R}$, it is said to be a **bounded interval**.

If it is not bounded in $\mathbb{R}$, it is said to be an **unbounded interval**.

We agree to say that $\emptyset$ is both an open interval and a closed interval.

We also agree to say that $\mathbb{R}$ is both an open interval and a closed interval.

8. **Example (2).**

Very often a predicate, say, $P(x)$, with variable x, used as the 'selection criterion' in the application of the Method of Specification is of the form

'*there exists some blah-blah-blah such that bleh-bleh-bleh*'

in which the variable x sits inside '*bleh-bleh-bleh*'.

(a) The predicate (with variable x) given by

'*there exists some* $n \in \mathbb{N}$ *such that* $x = n^2$'

is converted into a true statement exactly when any one square integers $(0, 1, 4, 9, 16, \cdots)$ is substituted into the variable x.

Hence

$$\{x \mid \text{there exists some } n \in \mathbb{N} \text{ such that } x = n^2\}$$

is the same as the set of all square integers '$\{0, 1, 4, 9, 16, \cdots\}$'.

The use of the Method of Specification spares us the trouble of having to clarifying what we mean by the dot-dot-dot's in '$0, 1, 4, 9, 16, \cdots$'.

Remark on short-hand.

When we allow the predicate

‘*there exists some* $n \in \mathbb{N}$ *such that* $x = n^2$’

to be presented as

‘$x = n^2$ *for some* $n \in \mathbb{N}$’,

we may present the set

$$\{x \mid \text{there exists some } n \in \mathbb{N} \text{ such that } x = n^2\}$$

as

$$\{x \mid x = n^2 \text{ for some } n \in \mathbb{N}\}.$$

We may ‘abbreviate’ the last expression as

$$\{n^2 \mid n \in \mathbb{N}\}.$$

This ‘short-hand’ is visually appealing as it suggests that the set concerned is constructed by collecting those and only those ‘numbers of the form n^2’ obtained ‘when n runs through all possible natural numbers’.

(b) The predicate (with variable x) given by

'*there exists some* $n \in \mathbb{Z}$ *such that* $x = 2n$'

is converted into a true statement exactly when any one even integers $(0, \pm 2, \pm 4, \pm 6, \cdots)$ is substituted into the variable x.

Hence:

i. $\{x \mid \text{there exists some } n \in \mathbb{Z} \text{ such that } x = 2n\}$ is the same as the set of all even integers

$$\text{`}\{\cdots, -6, -4, -2, 0, 2, 4, 6, \cdots\}\text{'}.$$

ii. $\{x \in \mathbb{N} : \text{there exists some } n \in \mathbb{Z} \text{ such that } x = 2n\}$ is the same as the set of all non-negative integers

$$\text{`}\{0, 2, 4, 6, 8, \cdots\}\text{'}.$$

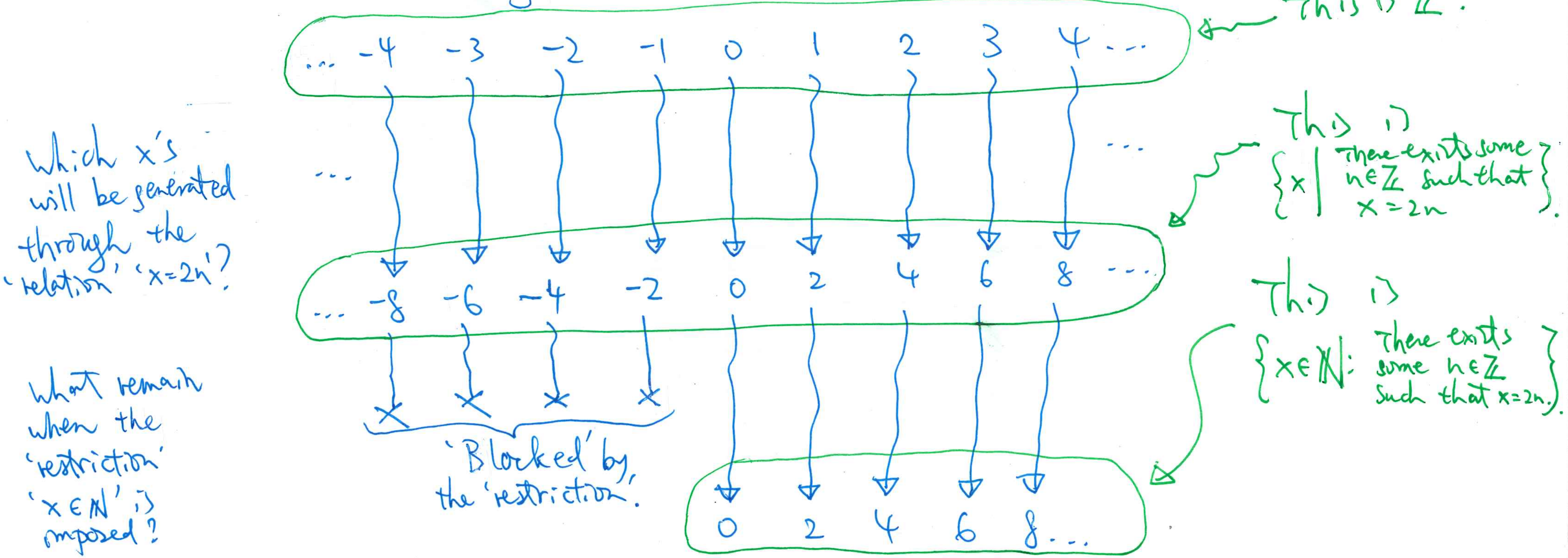
How to visualize

$$\{x \in \mathbb{N} : \text{there exits some } n \in \mathbb{Z} \text{ such that } x = 2n\}$$

as

$$`\{0, 2, 4, 6, 8, \ldots\}' ?$$

Amongst which objects does n 'vary'?



(c) The predicate (with variable x) given by

'*there exists some* $n \in \mathbb{Z}$ *such that* $x = n^3$'

is converted into a true statement exactly when any one even integers $(0, \pm 1, \pm 8, \pm 27, \cdots)$ is substituted into the variable x.

Hence:

i. $\{x \mid \text{there exists some } n \in \mathbb{Z} \text{ such that } x = n^3\}$ is the same as the set of all cube integers

$$\text{`}\{\cdots, -27, -8, -1, 0, 1, 8, 27, \cdots\}\text{'}.$$

ii. $\{x \in \mathbb{N} : \text{there exists some } n \in \mathbb{Z} \text{ such that } x = n^3\}$ is the same as the set of all non-negative cube integers

$$\text{`}\{0, 1, 8, 27, 64, \cdots\}\text{'}.$$

(d) The predicate (with variable x) given by

'*there exist some* $m, n \in \mathbb{N}$ *such that* $x = 2^m \cdot 3^n$'

is converted into a true statement exactly when any number which can be factorized as a product of 2's and 3's is substituted into the variable x.

Hence

$$\{x \mid \text{there exist some } m, n \in \mathbb{N} \text{ such that } x = 2^m \cdot 3^n\}$$

is the same as the set

$$\text{`}\{1, 2, 3, 4, 6, 8, 9, 12, 18, 27, \cdots\}\text{'}.$$

(e) The predicate (with variable z) given by

'*there exist some* $n \in \mathbb{N}$ *such that* $|z| = n$'

is converted into a true statement exactly when any complex number whose modulus is a natural number is substituted into the variable z.

Hence

$$\{z \in \mathbb{C} : \text{there exist some } n \in \mathbb{N} \text{ such that } |z| = n\}$$

is the same as the set

$$\{z \in \mathbb{C} : |z| \in \mathbb{N}\}.$$

It is the 'union' of all circles with centre at 0 and with radius being a natural number.

(f) Let H be a $(p \times q)$-matrix.

The predicate (with variable $\mathbf{y}$) given by

'*there exist some* $\mathbf{t} \in \mathbb{R}^q$ *such that* $\mathbf{y} = H\mathbf{t}$'

is converted into a true statement exactly when a vector in $\mathbb{R}^p$ which 'can be expressed as $H\mathbf{t}$ (for some appropriate vector $\mathbf{t}$)' is substituted into the variable $\mathbf{y}$.

The set $\{\mathbf{y} \in \mathbb{R}^p : \text{ there exist some } \mathbf{t} \in \mathbb{R}^q \text{ such that } \mathbf{y} = H\mathbf{t}\}$ is the **column space** of the matrix H, as introduced in your *linear algebra* course.

Short-hand.

We write this set as

$$\{\mathbf{y} \in \mathbb{R}^p : \mathbf{y} = H\mathbf{t} \text{ for some } \mathbf{t} \in \mathbb{R}^q\},$$

or even as

$$\{H\mathbf{t} \mid \mathbf{t} \in \mathbb{R}^q\}.$$

This 'short-hand' is visually appealing as it suggests that the set concerned is constructed by collecting those and only those 'vectors of the form $H\mathbf{t}$' obtained 'when $\mathbf{t}$ runs through all possible elements of $\mathbb{R}^q$'.

(g) Let $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ be vectors in $\mathbb{R}^n$.

The predicate (with variable $\mathbf{y}$) given by

'*there exist some* $\alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}$ *such that* $\mathbf{y} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k$'

is converted into a true statement exactly when a vector in $\mathbb{R}^n$ which 'can be expressed as a linear combination of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$' is substituted into the variable $\mathbf{y}$.

The set $\left\{ \mathbf{y} \in \mathbb{R}^n : \begin{array}{l} \text{there exist some } \alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R} \\ \text{such that } \mathbf{y} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k \end{array} \right\}$ is the **span** of the vectors $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$, as introduced in your *linear algebra* course.

Short-hand.

We write this set as

$$\left\{ \mathbf{y} \in \mathbb{R}^n : \begin{array}{c} \mathbf{y} = \alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k \\ \text{for some } \alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R} \end{array} \right\},$$

or even as

$$\{\alpha_1\mathbf{u}_1 + \alpha_2\mathbf{u}_2 + \cdots + \alpha_k\mathbf{u}_k \mid \alpha_1, \alpha_2, \cdots, \alpha_k \in \mathbb{R}\}.$$

This 'short-hand' is visually appealing as it suggests that the set concerned is constructed by collecting all possible linear combinations of $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_k$ in $\mathbb{R}^n$.

9. **'Solving an equation/inequality', understood as finding an 'explicit presentation' of its solution set.**

   To 'solve an equation/inequality' is the same as to determine all its solutions and find a way to present them explicitly.

   Very often such an equation/inequality has (infinitely) many solutions, and no one specific solution is privileged over another.

   This prompts us to present the collection of all solutions for such an equation/inequality as a set constructed by the method of specification, in which the 'selection criterion, used in the application of this method gives an explicit description of all individual solutions of the equation/inequality.

Illustrations.

(a) The solutions of the equation $x^2 - 3x + 2 = 0$ with real unknown x are exactly $1, 2$.

We may present the content of the above statement in the form of the set equality

$$\{x \in \mathbb{R} : x^2 - 3x + 2 = 0\} = \{1, 2\}.$$

The left-hand side is the solution set of the equation $x^2 - 3x + 2 = 0$ with real unknown x.
The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

(b) The solutions of the equation $x^2 + 4 = 0$ with complex unknown x are exactly $2i, -2i$.

We may present the content of the above statement in the form of the set equality

$$\{x \in \mathbb{C} : x^2 + 4 = 0\} = \{2i, -2i\}.$$

The left-hand side is the solution set of the equation $x^2 + 4 = 0$ with complex unknown x.
The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

(c) There is no solution for the equation $x^2 + 4 = 0$ with real unknown x.

We may present the content of the above statement in the form of the set equality

$$\{x \in \mathbb{R} : x^2 + 4 = 0\} = \emptyset.$$

The left-hand side is the solution set of the equation $x^2 + 4 = 0$ with real unknown x.

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

(d) The solutions of the inequality $x^2 - 3x + 2 < 0$ with real unknown x are exactly the real numbers strictly between 1 and 2.

We may present the content of the above statement in the form of the set equality

$$\{x \in \mathbb{R} : x^2 - 3x + 2 < 0\} = \{x \in \mathbb{R} : 1 < x < 2\}.$$

The left-hand side is the solution set of the equation $x^2 - 3x + 2 < 0$ with real unknown x.

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

(e) Suppose α is a real number.

Then α is a solution $\sin(x) = \frac{1}{2}$ with real unknown x iff

$$\text{there exists some } n \in \mathbb{Z} \text{ such that } \alpha = n\pi + (-1)^n \cdot \frac{\pi}{6}.$$

We may present the content of the above statement in the form of the set equality

$$\left\{ x \in \mathbb{R} : \sin(x) = \frac{1}{2} \right\} = \left\{ x \in \mathbb{R} : \begin{array}{l} \text{There exist some } n \in \mathbb{Z} \text{ such that} \\ n\pi + (-1)^n \cdot \frac{\pi}{6} \end{array} \right\}.$$

The left-hand side is the solution set of the equation $\sin(x) = \frac{1}{2}$ with real unknown x.

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

(f) Define $A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 6 \\ 6 \end{bmatrix}$.

Suppose $\mathbf{v}$ is a vector in $\mathbb{R}^3$.

Then $\mathbf{v}$ is a solution of the equation $A\mathbf{x} = \mathbf{b}$ with unknown vector $\mathbf{x}$ in $\mathbb{R}^3$ iff

$$\text{there exists some } t \in \mathbb{R} \text{ such that } \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

We may present the content of the above statement in the form of the set equality

$$\begin{aligned} &\{\mathbf{x} \in \mathbb{R}^3 : A\mathbf{x} = \mathbf{b}\} \\ = &\left\{ \mathbf{x} \in \mathbb{R}^3 : \text{ There exists some } t \in \mathbb{R} \text{ such that } \mathbf{x} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

The left-hand side is the solution set of the equation $A\mathbf{x} = \mathbf{b}$ with real vector unknown $\mathbf{x}$.

The right-hand side is the *same* set, in which every solution of this equation is presented explicitly.

10. **Example (3).**

(a) The predicate

$$\text{`}x \neq x\text{'}$$

yields a false statement, no matter which object is substituted into it.

For this reason, the set

$$\{x \mid x \neq x\}$$

is the empty set.

Also, for any set A, the set

$$\{x \in A : x \neq x\}$$

is the empty set.

(b) Consider the predicate

$$\text{`}\textit{for any } n \in \mathbb{N},\ x = n^2\text{'},$$

in which the variable is x.

This predicate yields a false statement no matter which object is substituted into x.

Hence the set

$$\{x \mid \text{ for any } n \in \mathbb{N}, x = n^2\}$$

is the empty set.

Remark.

Contrast this set with

$$\{x \mid \text{ there exists some } n \in \mathbb{N} \text{ such that } x = n^2\}.$$

The latter is the set of all square integers.