1. **Mathematical statements**.

   A **mathematical statement** *is a sentence with mathematical content (or several inter-related sentences which can be condensed into one through the appropriate use of clauses), for which it is meaningful to say it is true or it is false.*

   It is with mathematical statements that mathematicians formulate their ideas and present mathematical theories at a professional level.

2. **Examples of mathematical statements from school maths and previous courses.**

   You must have encountered many mathematical statements in your mathematics classes or in your mathematics textbooks. Many of the most important statements that you have seen are '*theorems*'. Some are '*definitions*'.

   Each of these passages are examples of mathematical statements:—

   (a) $1 + 1 = 2$.

   (b) $1 + 1 = 3$.

   Remark. This is a mathematical statement, despite the fact that it is a false (mathematical) statement.

   (c) **Pythagoras' Theorem.**

   Let $\triangle ABC$ be a triangle. Suppose $\angle ACB$ is a right angle. Then $AB^2 = AC^2 + BC^2$.

   (d) **Converse of Pythagoras' Theorem.**

   Let $\triangle ABC$ be a triangle. Suppose $AB^2 = AC^2 + BC^2$. Then $\angle ACB$ is a right angle.

   (e) **Definition for the notions of differentiability and derivative.**

   Let f be a real-valued function of one real variable, well-defined on some open interval I. Let c be a point in I. We say f is differentiable at c if the limit $\lim\limits_{t \to 0} \dfrac{f(c+t) - f(c)}{t}$ exists.

   Where this limit indeed exists, its value is called the derivative of f at c, and is denoted by $f'(c)$.

   (f) **Mean-Value Theorem.**

   Let a, b be real numbers, with $a < b$, and f be a real-valued function of one real variable, well-defined on $[a, b]$. Suppose f is continuous on $[a, b]$ and f is differentiable on (a, b). Then there exists some $c \in (a, b)$ such that $\dfrac{f(b) - f(a)}{b - a} = f'(c)$.

   (g) **Definition for the notions of null space.**

   Let A be an $(m \times n)$-matrix with real entries. The null space of A is defined to be the set $\{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}_m\}$.

   (h) **Theorem on the re-formulation of linear independence in terms of null space.**

   Suppose U is an $(m \times n)$-matrix with real entries, whose columns from left to right are denoted $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$. Then the zero vector in $\mathbb{R}^n$ is the only vector in the null space of U iff $\mathbf{u}_1, \mathbf{u}_2, \cdots, \mathbf{u}_n$ are linearly independent.

3. **Special features in mathematical statements, in contrast to sentences in everyday language.**

   Mathematical statements allow no room for inaccuracies, no room for ambiguities, no room for metaphors.

   (a) *Words or phrases within a statement with specific mathematical content have to be explained carefully (usually prior to the formulation of the statement).*

   Examples of words/phrases that would require precise definitions.

   i. irrational number.
   ii. prime number.
   iii. root of a polynomial.
   iv. geometric progression.
   v. derivative.
   vi. definite integral.
   vii. linear combination.
   viii. invertible matrices.

   (b) *Some words which appear in everyday language may have very specific meaning.*

   Examples (related to logic).

i. and. ii. or. iii. then. iv. some. v. every.

Examples in general.

i. continuous. ii. smooth. iii. open. iv. closed. v. connected. vi. increasing.

(c) *The inclusion or the exclusion of one or several words, or simply a change of one word will change the meaning of the statement.*

Examples.

i. The statements $(\sharp_1), (\sharp_2)$ below differ from each other by one word; hence they are totally different statements in terms of content.

$(\sharp_1)$ *There is at least one even integer.*

$(\sharp_2)$ *There is at most one even integer.*

(The statement $(\sharp_1)$ is a true statement. The statement $(\sharp_2)$ is a false statement.)

ii. The statements $(\flat_1), (\flat_2)$ below differ from each other by one word; hence they are totally different statements in terms of content.

$(\flat_1)$ *The number 1 is a real solution of the equation* $x^4 - 1 = 0$.

$(\flat_2)$ *The number 1 is the real solution of the equation* $x^4 - 1 = 0$.

(The statement $(\flat_1)$ is a true statement. The statement $(\flat_2)$ is a false statement, because the word '*the*' means '*the only*'.)

(d) *Very often ordering of words/symbols and grouping of words/symbols matter in the meaning of a statement.*

Examples.

i. The statements $(\star_1), (\star_2)$ below are made up of the same words, but put in different order. Exactly because of the ordering of words, the statements are totally different in terms of content:

$(\star_1)$ *Not every integer is a prime number.*

$(\star_2)$ *Every integer is not a prime number.*

(The statement $(\star_1)$ is a true statement. The statement $(\star_2)$ is a false statement.)

ii. The statements $(\dagger_1), (\dagger_2)$ below are made up of the same words, but put in different order. Exactly because of the ordering of words, the statements are totally different in terms of content:

$(\dagger_1)$ *Let x be a real number. Suppose $2 < x < 3$. Then $x^2 - 5x + 4 < 0$.*

$(\dagger_2)$ *Let x be a real number. Suppose $x^2 - 5x + 4 < 0$. Then $2 < x < 3$.*

(The statement $(\dagger_1)$ is a true statement. The statement $(\dagger_2)$ is a false statement.)

iii. The statements $(*_1), (*_2)$ below are made up of the same words, but put in different order. Exactly because of the ordering of words, the statements are totally different in terms of content:

$(*_1)$ *For every real number α, there is an integer n so that '$\alpha < n$' is true.*

$(*_2)$ *There is an integer n so that for every real number α, '$\alpha < n$' is true.*

(The statement $(*_1)$ is a true statement, and a useful statement in mathematics. The statement $(*_2)$ is a false statement.)

iv. The statements $(\ddagger_1), (\ddagger_2)$ below are made up of the same words, but grouped in different ways by parentheses. Exactly because of the ways of grouping, the statements are totally different in terms of content:

$(\ddagger_1)$ *Let x be a real number. Suppose $x(x-1)(x-2) > 0$. Then ($x < 1$ and $x > 0$) or $x > 2$.*

$(\ddagger_2)$ *Let x be a real number. Suppose $x(x-1)(x-2) > 0$. Then $x < 1$ and ($x > 0$ or $x > 2$).*

(The statement $(\ddagger_1)$ is a true statement, and a useful statement in mathematics. The statement $(\ddagger_2)$ is a false statement.)

4. **Definitions as statements.**

A **definition** *is a mathematical statement which informs the reader of the mathematical meaning of a specific mathematical term/phrase. We do not have to justify whether a definition is true; they are by default regarded as true statements.*

(There are, however, good or bad definitions. The good ones are those which are 'mathematically useful'; this involves mathematical and philosophical judgement.)

Very often, a statement which gives the definition of a specific attribute that applies to some specific type of objects looks like the following:—

Let/Suppose x be amongst blah-blah-blah.

We say x is so-and-so if the statement 'bleh-bleh-bleh-bleh-bleh-bleh' (about x) holds.

The word/phrase inside 'so-and-so' is the attribute to be defined that applies to all objects amongst 'blah-blah-blah'.

The statement 'bleh-bleh-bleh-bleh-bleh-bleh'can be called the 'defining condition' for the attribute 'so-and-so'. It informs us exactly the criterion on which an object amongst 'blah-blah-blah' may be given the attribute 'so-and-so'.

The 'defining condition' is (and should be) formulated in terms of words/phrases which have been defined earlier (and so understood already).

The word '*if*' used between '*x is so-and-so*' and *the statement 'bleh-bleh-bleh-bleh-bleh-bleh' (about x) holds* should be understood as 'if and only if'.

Examples of definitions.

(a) **Definition for the notion of arithmetic progression.**

Let $\{a_n\}_{n=0}^{\infty}$ be an infinite sequence in $\mathbb{C}$. The infinite sequence $\{a_n\}_{n=0}^{\infty}$ is said to be an **arithmetic progression** *if the statement* (AP) *holds:*

(AP) *There exists some $d \in \mathbb{C}$ such that for any $n \in \mathbb{N}$, $a_{n+1} - a_n = d$.*

Here:—

i. the attribute 'being an arithmetic progression', in the context '*$\{a_n\}_{n=0}^{\infty}$ is an arithmetic progression*', is to be defined that applies to the objects 'infinite sequences of complex numbers', and

ii. the 'defining condition' is the statement (AP).

(b) **Definition for the notion of divisibility.**

Let u, v be integers.

u is said to be **divisible** *by v if there exists some integer k such that $u = kv$.*

Here:—

i. the attribute 'divisible', in the context '*u is divisible by v*' is to be defined that applies to integers, and

ii. the 'defining condition' is the statement '*there exists some integer k such that $u = kv$*'.

5. **Conjectures and theorems as statements.**

In a mathematical theory, some mathematical statements which are not definitions are singled out because the content of such a statement is important within the theory.

When such a statement is yet to be proved or dis-proved (on the basis of what is regarded to be more fundamental), we call it a **conjecture**.

When such a statement has been proved, it is called a **theorem** (or, depending on the context, called a proposition or called a lemma).

(*This is a study tip.* Within a mathematical theory, the most important conjectures/theorems are given names. Very often such a statement is simply named after the mathematicians who discovered the statement, or whom others believe to have discovered the statement. However, be aware that various unrelated statements can be named after the same mathematician.)

Examples.

(a) **Fermat's Little Theorem.**

Suppose p is a positive prime number, and x is an integer. Then $x^p - x$ is divisible by p.

(b) **Fermat's Last Theorem.**

Let n be an integer greater than 2. Suppose x, y, z are positive integers. Then $x^n + y^n \neq z^n$.

Remark. Despite the name of this statement, the first successful proof of the statement appeared only in the mid-1990's. For more than 300 years this statement had only been a conjecture.

6. **Conditional, assumption, conclusion, converse.**

With few exceptions, all conjectures/theorems that you have encountered, or will ever encounter, can be presented as 'conditionals', like the statement ($\star$):

($\star$) '**Let** *blah-blah-blah.* **Suppose** *bleh-bleh-bleh.* **Then** *blih-blih-blih.*'

($\star$) can be condensed into the form:

‘**For any** *blah-blah-blah,* **if** *bleh-bleh-bleh,* **then** *blih-blih-blih.*’

The content within ‘*blah-blah-blah*, *bleh-bleh-bleh*’ collectively is often referred to as the ‘assumption part’ of the statement.

The content within ‘*blih-blih-blih*’ is often referred to as the ‘conclusion part’ of the statement.

Interchanging the positions of ‘*bleh-bleh-bleh*’ and ‘*blih-blih-blih*’, we obtain another statement ($\star\star$), called the **converse** of the statement ($\star$):

($\star\star$) ‘**Let** *blah-blah-blah.* **Suppose** *blih-blih-blih.* **Then** *bleh-bleh-bleh.*’

($\star\star$) may be condensed into:

‘**For any** *blah-blah-blah,* **if** *blih-blih-blih,* **then** *bleh-bleh-bleh.*’

We emphasize that the statement ($\star$) and its converse ($\star\star$) are distinct in terms of content. It can happen that both are true, or that both are false, or that one of them is true and the other is false.

When it happens that both ($\star$) and ($\star\star$) are true, the content of both statements may be collectively presented as:

($\star\star\star$) ‘**Let/Suppose** *blah-blah-blah.* **Then** *bleh-bleh-bleh* **iff** *blih-blih-blih.*’

It may be condensed into:

‘**For any** *blah-blah-blah, bleh-bleh-bleh* **iff** *blih-blih-blih.*’

Where it is appropriate, the content ‘*blah-blah-blah*’ may be split in such a way that

Or it may be presented in a more ‘expansive’ way:

‘**Let/Suppose** *blah-blah-blah.* **Then** *the statements* ($\dagger_1$), ($\dagger_2$) *below are* **logically equivalent***:*

($\dagger_1$) ‘*bleh-bleh-bleh.*’

($\dagger_2$) ‘*blih-blih-blih.*’

Examples.

(a) ($\star$) **Pythagoras’ Theorem. (Proposition 47, Book I of Euclid’s Elements.)**
Let $\triangle ABC$ *be a triangle. Suppose* $\angle ACB$ *is a right angle. Then* $AB^2 = AC^2 + BC^2$*.*

($\star\star$) **Converse of Pythagoras’ Theorem. (Proposition 48, Book I of Euclid’s Elements.)**
Let $\triangle ABC$ *be a triangle. Suppose* $AB^2 = AC^2 + BC^2$*. Then* $\angle ACB$ *is a right angle.*

Both Pythagoras’ Theorem and its converse are true.

The two statements may be combined into one statement:

Suppose $\triangle ABC$ *be a triangle. Then* $\angle ACB$ *is a right angle iff* $AB^2 = AC^2 + BC^2$*.*

(b) ($\star$) *Let* $\triangle ABC$ *be a triangle. Suppose* $\triangle ABC$ *is equilateral. Then* $\triangle ABC$ *is isosceles.*

($\star\star$) *Let* $\triangle ABC$ *be a triangle. Suppose* $\triangle ABC$ *is isosceles. Then* $\triangle ABC$ *is equilateral.*

The statement ($\star$) is true. Its converse ($\star\star$) is false.

(c) ($\star$) *Let* $\triangle ABC$ *be a triangle. Suppose* $\angle ABC$ *is a right angle. Then* $\triangle ABC$ *is equilateral.*

($\star\star$) *Let* $\triangle ABC$ *be a triangle. Suppose* $\triangle ABC$ *is equilateral. Then* $\angle ABC$ *is a right angle.*

The statement ($\star$) is false. Its converse ($\star\star$) is also false.

7. **Axioms as statements.**

Dependent on the scope of a mathematical theory, a few mathematical statements can be regarded to be so fundamental that their validity is assumed without proof. Such statements are called **axioms**.

Examples of axioms in Euclidean geometry.

(a) **‘First Postulate’ in Euclid’s Elements.**
There is (at least) one straight line joining any two given points in the plane.

(b) **‘Fifth Postulate’ in Euclid’s Elements.**
Let ℓ, m, n *be infinite straight lines. Suppose that on the same side of* ℓ*, the sum of the interior angle subtended by* ℓ *and* m*, and of the interior angle subtended by* ℓ *and* n*, add up to less than two right angles. Then* m, n *(when produced on that same side of* ℓ*) will intersect each other.*

Examples of axioms in mathematical analysis.

(a) **Well-ordering Principle for integers.**

Let S be a subset of $\mathbb{N}$. Suppose S is non-empty. Then S has a least element.

(b) **Least-upper-bound Axiom.**

Let T be a non-empty subset of $\mathbb{R}$. Suppose T is bounded above in $\mathbb{R}$. Then T has a least upper bound.

Be careful: the axioms in one mathematical theory can become theorems in another theory, if the latter is meant to serve as a foundation for the former.

8. **Proofs.**

Roughly speaking, a proof for a statement is a passage of several inter-related statements which provides the explanation for the statement concerned, based on other statements already known to be true.

The simplest forms of proofs are 'direct proof' and 'proof-by-contradiction'.

Consider the statement ($\star$), which reads:

($\star$) '**Let** *blah-blah-blah.* **Suppose** *bleh-bleh-bleh.* **Then** *blih-blih-blih.*'

(a) A 'direct proof' for ($\star$) is of the form:

Let *blah-blah-blah.* Suppose *bleh-bleh-bleh.*
Then *bloh-bloh-bloh-bloh-bloh-bloh.*
⋮ ⋮ ⋮ ⋮ ⋮
Therefore *bluh-bluh-bluh-bluh-bluh-bluh.*
Hence *blih-blih-blih.*

The 'assumption part' of the statement ($\star$) appears at the top of this passage, and the 'conclusion part' of the statement ($\star$) at the bottom at the passage. What is written at each 'intermediate step' (from '*bloh-bloh-bloh-bloh-bloh-bloh*' to '*bluh-bluh-bluh-bluh-bluh-bluh*') will have, as its justification, something already established within the passage, or something known to be 'true in general'.

For concrete examples of this type of arguments, refer to the handouts *From simple inequalities to basic properties of the reals* and *Basic results on divisibility.*

(b) An argument given by the 'proof-by-contradiction' method for ($\star$) is of the form:

Let *blah-blah-blah.* Suppose *bleh-bleh-bleh.*
Further suppose that it was not true that '*blih-blih-blih*' held.
Then '*bloh-bloh-bloh-bloh-bloh-bloh*' would hold.
⋮ ⋮ ⋮ ⋮ ⋮
Therefore '*bluh-bluh-bluh-bluh-bluh-bluh*' would hold.
However '*bluh-bluh-bluh-bluh-bluh-bluh* is (known/assumed to be) not true.
Now '*bluh-bluh-bluh-bluh-bluh-bluh*' would be simultaneously true and not true. Contradiction arises.
It follows that, in the first place, '*blih-blih-blih*' holds.

The 'assumption part' of the statement ($\star$) appears at the top of this passage, and for the moment, for the sake of argument, it is further supposed that the 'conclusion part' of the statement ($\star$) fails to hold.

Starting from this premise, we look for something 'ridiculously wrong', called a contradiction, out of the combination of what has been supposed and what has been further supposed.

Then we will be made to conclude that under the assumption in ($\star$), the desired conclusion in ($\star$) will be 'forced' to hold.

What is written at each 'intermediate step' (from '*bloh-bloh-bloh-bloh-bloh-bloh* would hold' to '*bluh-bluh-bluh-bluh-bluh-bluh* would hold') in the 'main body of the argument' will have, as its justification, something assumed at the top, or already established within this passage, or something known to be 'true in general'.

For concrete examples of this type of arguments, refer to the handout *Rationals and irrationals.*