MATH1050 Answers to Examples: De Moivre's Theorem and Roots of unity.

1. (a) $\sqrt[4]{2}(\cos(\frac{\pi}{8}) + i\sin(\frac{\pi}{8}))$, $\sqrt[4]{2}(\cos(-\frac{7\pi}{8}) + i\sin(-\frac{7\pi}{8}))$.

   (b) $\sqrt[6]{2}(\cos(\frac{\pi}{12}) + i\sin(\frac{\pi}{12}))$, $\sqrt[6]{2}(\cos(\frac{3\pi}{4}) + i\sin(\frac{3\pi}{4}))$, $\sqrt[6]{2}(\cos(-\frac{7\pi}{12}) + i\sin(-\frac{7\pi}{12}))$.

   (c) $\sqrt[8]{2}(\cos(\frac{\pi}{16}) + i\sin(\frac{\pi}{16}))$, $\sqrt[8]{2}(\cos(\frac{9\pi}{16}) + i\sin(\frac{9\pi}{16}))$, $\sqrt[8]{2}(\cos(-\frac{15\pi}{16}) + i\sin(-\frac{16\pi}{16}))$, $\sqrt[8]{2}(\cos(-\frac{7\pi}{16}) + i\sin(-\frac{7\pi}{16}))$.

2. (a) The solutions of the equation $\cos(x) = \frac{1}{\sqrt{2}}$ are described by '$x = \pm\frac{\pi}{4} + 2N\pi$ for some integer N'.

   (b) i. ——

   ii. $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^m + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^m = \zeta^m + \bar{\zeta}^m = 2r^m\cos(m\theta) = 2\cos(m\theta)$.

   iii. $\left(\frac{1}{\sqrt{2}} + \frac{i}{\sqrt{2}}\right)^m + \left(\frac{1}{\sqrt{2}} - \frac{i}{\sqrt{2}}\right)^m = \sqrt{2}$ iff ($m = \pm 1 + 8N$ for some integer N).

3. (a) $(1+i)^p - (1-i)^p = 0$ iff ($p = 4N$ for some $N \in \mathbb{Z}$).

   (b) 2.

4. (a) $\omega = \cos(\frac{2\pi}{3}) + i\sin(\frac{2\pi}{3})$ or $\omega = \cos(-\frac{2\pi}{3}) + i\sin(-\frac{2\pi}{3})$.

   (b) i. ——

   ii. $\alpha^3 = 1 = \beta^3$.

   iii. $1, -1, 2$.

5. (a) i. ——

   ii. $\frac{(z^2 - 1/z^2)i}{z^2 + 1/z^2} = -\sqrt{3}$ iff ($z = i^M\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$ for some $M \in \mathbb{Z}$).

   (b) i. $\cos(\frac{\pi}{3}) + i\sin(\frac{\pi}{3})$, $\cos(-\frac{\pi}{3}) + i\sin(-\frac{\pi}{3})$.

   ii. A. $x^2 - 2x + 1$. B. $x^2 + x + 1$. C. $x^2 + x + 1$.

6. (a) *Hint.*

   You need De Moivre's Theorem. Also recall the for any $\eta \in \mathbb{C}$, we have $2\mathsf{Re}(\eta) = \eta + \bar{\eta}$ and $2i\mathsf{Im}(\eta) = \eta - \bar{\eta}$.

   (b) *Hint.* You need the Binomial Theorem.

7. (a) ——

   (b) i. $\cos^7(\theta) = \frac{1}{2^6}(\cos(7\theta) + 7\cos(5\theta) + 21\cos(3\theta) + 35\cos(\theta))$.

   $\sin^7(\theta) = \frac{1}{2^6}(-\sin(7\theta) + 7\sin(5\theta) - 21\sin(3\theta) + 35\sin(\theta))$.

   ii. $\cos^8(\theta) = \frac{1}{2^7}(\cos(8\theta) + 8\cos(6\theta) + 28\cos(4\theta) + 56\cos(2\theta) + 35)$.

   $\sin^8(\theta) = \frac{1}{2^7}(\cos(8\theta) - 8\cos(6\theta) + 28\cos(4\theta) - 56\cos(2\theta) + 35)$.

8. ——

9. (a) ——

   (b) ——

   (c) i. $\omega = \frac{-1+\sqrt{5}}{4} + \frac{\sqrt{10+2\sqrt{5}}}{4}i$.

   ii. $\omega^2 = \frac{-1-\sqrt{5}}{4} + \frac{\sqrt{10-2\sqrt{5}}}{4}i$.

   iii. $\omega^3 = \overline{\omega^2} = \frac{-1-\sqrt{5}}{4} - \frac{\sqrt{10-2\sqrt{5}}}{4}i$, and $\omega^4 = \bar{\omega} = \frac{-1+\sqrt{5}}{4} - \frac{\sqrt{10+2\sqrt{5}}}{4}i$.