MATH1050 Examples: De Moivre's Theorem and roots of unity.

1. Let $\zeta = 1 + i$. Find:

   (a) the square roots of ζ,

   (b) the cube roots of ζ,

   (c) the quartic roots of ζ.

   Leave your answer in polar form.

2. (a) Solve for all real solutions of the equation $\cos(x) = \dfrac{1}{\sqrt{2}}$ with unknown x.

   (b) Let m be a positive integer.

      i. Let $r, \theta \in \mathbb{R}$, and $z = r(\cos(\theta) + i\sin(\theta))$. Show that $z^m + \bar{z}^m = 2r^m \cos(m\theta)$.

      ii. Express $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^m + \left(\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}\right)^m$ in the form $A\cos(B\theta)$, in which A, B are real numbers, possibly dependent on m.

      iii. Hence or otherwise, find all possible values of m for which $\left(\dfrac{1}{\sqrt{2}} + \dfrac{i}{\sqrt{2}}\right)^m + \left(\dfrac{1}{\sqrt{2}} - \dfrac{i}{\sqrt{2}}\right)^m = \sqrt{2}$.

3. (a) Let p be a positive integer. Find all possible values of p for which $(1+i)^p - (1-i)^p = 0$.

   (b) Hence, or otherwise, find the value of $\dfrac{(1+i)^{4k+1}}{(1-i)^{4k-1}}$, where k is a positive integer.

4. (a) Let θ be a real number. Suppose θ is not an integral multiple of π.

   Let $\omega = \cos(\theta) + i\sin(\theta)$. Suppose $\omega^2 - 2\bar{\omega} + \dfrac{1}{\omega}$ is real. Find the two possible values of ω.

   (b) Denote by α, β the two values of ω obtained in the previous part.

      i. Prove that $\alpha^2 = \beta$ and $\beta^2 = \alpha$.

      ii. Find the respective values of α^3 and β^3.

      iii. Suppose n is an integer. Find all possible values of $\alpha^n + \beta^n$.

5. (a) i. Let $\theta \in \mathbb{R}$, and $\omega = \cos(\theta) + i\sin(\theta)$. Let n be a positive integer.

      Prove that $\omega^n + \dfrac{1}{\omega^n} = 2\cos(n\theta)$ and $\omega^n - \dfrac{1}{\omega^n} = 2i\sin(n\theta)$.

      ii. Hence find all complex numbers which satisfies $\dfrac{(z^2 - 1/z^2)i}{z^2 + 1/z^2} = -\sqrt{3}$.

   (b) Consider the polynomial $f(x) = x^2 - x + 1$.

      i. Express, in polar form, the roots of $f(x)$.

      ii. Let α, β be the roots of $f(x)$. Let k be an integer. Find the quadratic polynomial with leading coefficient 1 whose roots are $\left(\dfrac{\alpha}{\beta}\right)^k$ and $\left(\dfrac{\beta}{\alpha}\right)^k$ for the various scenarios below:

         A. $k = 3p$ for some $p \in \mathbb{Z}$.

         B. $k = 3q + 1$ for some $q \in \mathbb{Z}$.

         C. $k = 3r + 2$ for some $r \in \mathbb{Z}$.

6. Let $\theta \in \mathbb{R}$, and $z = \cos(\theta) + i\sin(\theta)$.

   (a) Let $m \in \mathbb{N}\backslash\{0\}$. Prove that $2\cos(m\theta) = z^m + \bar{z}^m$ and $2i\sin(m\theta) = z^m - \bar{z}^m$.

   (b) Let $n \in \mathbb{N}\backslash\{0\}$. Applying the result above, or otherwise, prove the statements below:

i. $\cos(2n\theta) = \sum_{j=0}^{n}(-1)^j \binom{2n}{2j} \cos^{2n-2j}(\theta)\sin^{2j}(\theta)$, *and*

$\sin(2n\theta) = \sum_{j=0}^{n-1}(-1)^j \binom{2n}{2j+1} \cos^{2n-2j-1}(\theta)\sin^{2j+1}(\theta)$.

ii. $\cos((2n+1)\theta) = \sum_{j=0}^{n}(-1)^j \binom{2n+1}{2j} \cos^{2n+1-2j}(\theta)\sin^{2j}(\theta)$, *and*

$\sin((2n+1)\theta) = \sum_{j=0}^{n}(-1)^j \binom{2n+1}{2j+1} \cos^{2n-2j}(\theta)\sin^{2j+1}(\theta)$.

7. Let $\theta \in \mathbb{R}$ and $z = \cos(\theta) + i\sin(\theta)$.

(a) Let $n \in \mathbb{N}\backslash\{0\}$. Prove that $2^n \cos^n(\theta) = \left(z + \frac{1}{z}\right)^n$ and $2^n i^n \sin^n(\theta) = \left(z - \frac{1}{z}\right)^n$.

(b) Hence, or otherwise, deduce the results below:

i. $\cos^7(\theta) = \frac{1}{2^C}\sum_{j=0}^{7} A_j \cos(j\theta)$ *and* $\sin^7(\theta) = \frac{1}{2^D}\sum_{j=0}^{7} B_j \sin(j\theta)$.

Here $A_0, A_1, A_2, ..., A_7$, $B_0, B_1, B_2, ..., B_7$, C, D are integers whose respective value you have to determine.

ii. $\cos^8(\theta) = \frac{1}{2^C}\sum_{j=0}^{8} A_j \cos(j\theta)$ *and* $\sin^8(\theta) = \frac{1}{2^D}\sum_{j=0}^{8} B_j \cos(j\theta)$.

Here $A_0, A_1, A_2, ..., A_8$, $B_0, B_1, B_2, ..., B_8$, C, D are integers whose respective value you have to determine.

Remark. Can you generalize the idea for the situation of $\cos^n(\theta)$, $\sin^n(\theta)$, in which n is an arbitrary positive integer?

8. Let $n \in \mathbb{N}$. Let α, θ be real numbers. Suppose $\sin(\theta/2) \neq 0$.

(a) Prove that

$$\sum_{k=0}^{n} \cos(\alpha + k\theta) = \frac{\sin((n+1)\theta/2)\cos(\alpha + n\theta/2)}{\sin(\theta/2)}, \text{ and } \sum_{k=0}^{n} \sin(\alpha + k\theta) = \frac{\sin((n+1)\theta/2)\sin(\alpha + n\theta/2)}{\sin(\theta/2)}.$$

(b) Suppose $n \geq 3$. Write $\varphi = \frac{2\pi}{n}$. Let $\omega_n = \cos(\varphi) + i\sin(\varphi)$. Let $\eta \in \mathbb{C}$. Suppose $|\eta| = 1$.

Prove that $\sum_{k=0}^{n-1} |\eta - {\omega_n}^k|^2 = 2n$.

Remark. Below is the geometric interpretation of the result. Consider the regular n-sided polygon with vertices $1, \omega_n, {\omega_n}^2, \cdots, {\omega_n}^{n-1}$. The sum of the squares of the lengths of the n chords joining respectively the n points of this polygon with any one point on the unit circle with centre 0 is the constant n.

9. Write $\theta = \frac{2\pi}{5}$, $\omega = \cos(\theta) + i\sin(\theta)$, $\sigma = \omega + \omega^4$, $\tau = \omega^2 + \omega^3$. Let $g(u)$ be the polynomial given by $g(u) = u^2 + u - 1$.

(a) Verify that $1 + \omega + \omega^2 + \omega^3 + \omega^4 = 0$.

(b) Verify that σ, τ are the distinct real roots of the polynomial $g(u)$.

(c) i. Prove that $\mathsf{Re}(\omega) = a + b\sqrt{M}$. Hence, or otherwise, find the value of ω.

Here a, b are rational numbers and M is an integer. You have to determine the explicit value of a, b, M.

ii. Prove that $\mathsf{Re}(\omega^2) = c + d\sqrt{N}$. Hence, or otherwise, find the value of ω^2.

Here c, d are rational numbers and N is an integer. You have to determine the explicit value of c, d, N.

iii. Find the respective values of ω^3, ω^4.